

**INF 1004 Mathematics 2**  
**Tutorial #10**

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## Question 1

Compute  $A^\top A$  for

$$A = \begin{bmatrix} 2 & -3 \\ -6 & -9 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 0 & 2 \\ 4 & 3 & 1 \end{bmatrix}$$

## My Solution

### Part 1

$$A = \begin{bmatrix} 2 & -3 \\ -6 & -9 \end{bmatrix}$$

$$A^\top = \begin{bmatrix} 2 & -6 \\ -3 & -9 \end{bmatrix}$$

$$\begin{aligned} A^\top A &= \begin{bmatrix} 2 & -6 \\ -3 & -9 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -6 & -9 \end{bmatrix} \\ &= \begin{bmatrix} 40 & 48 \\ 48 & 100 \end{bmatrix} \end{aligned}$$

### Part 2

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 0 & 2 \\ 4 & 3 & 1 \end{bmatrix}$$

$$A^\top = \begin{bmatrix} 0 & 2 & 4 \\ 1 & 0 & 3 \\ -1 & 2 & 1 \end{bmatrix}$$

$$\begin{aligned} A^\top A &= \begin{bmatrix} 0 & 2 & 4 \\ 1 & 0 & 3 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ 2 & 0 & 2 \\ 4 & 3 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 20 & 12 & 8 \\ 12 & 10 & 2 \\ 8 & 2 & 6 \end{bmatrix} \end{aligned}$$

## Question 2

Compute the inverse of

$$A_0 = \begin{bmatrix} -2 & -3 \\ -6 & -4 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 3 & 1 \\ -2 & 2 \end{bmatrix}$$

Use this inverses to solve

$$A_0 x = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$A_0 x = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

Note: it is not common to solve  $Ax = b$  using matrix inversion.

Reasons:

- $Ax = b$  can be solvable when  $A$  is not invertible.
- It is often slower / more costly

## My Solution

### Part 1

$$\begin{aligned} A^{-1} &\Rightarrow \left[ \begin{array}{cc|cc} -2 & -3 & 1 & 0 \\ -6 & -4 & 0 & 1 \end{array} \right] \\ -\frac{1}{2}\rho_1, -\frac{1}{4}\rho_2 &\Rightarrow \left[ \begin{array}{cc|cc} 1 & \frac{3}{2} & -\frac{1}{2} & 0 \\ \frac{3}{2} & 1 & 0 & -\frac{1}{4} \end{array} \right] \\ \frac{3}{2}\rho_1 - \rho_2 &\Rightarrow \left[ \begin{array}{cc|cc} 1 & \frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{5}{4} & -\frac{3}{4} & \frac{1}{4} \end{array} \right] \\ \frac{4}{5}\rho_2 &\Rightarrow \left[ \begin{array}{cc|cc} 1 & \frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{3}{5} & \frac{1}{5} \end{array} \right] \\ \rho_1 - \frac{3}{2}\rho_2 &\Rightarrow \left[ \begin{array}{cc|cc} 1 & 0 & \frac{2}{5} & -\frac{3}{10} \\ 0 & 1 & -\frac{3}{5} & \frac{1}{5} \end{array} \right] \end{aligned}$$

$$A^{-1} = \begin{bmatrix} \frac{2}{5} & -\frac{3}{10} \\ \frac{3}{5} & \frac{1}{10} \end{bmatrix}$$

$$\begin{aligned} A_0 x &= \begin{bmatrix} 3 \\ 1 \end{bmatrix} \\ A_0^{-1} A_0 x &= A_0^{-1} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \\ x &= \begin{bmatrix} \frac{2}{5} \\ \frac{3}{5} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0.9 \\ -1.6 \end{bmatrix} \end{aligned}$$

**Part 2**

$$A^{-1} \Rightarrow \left[ \begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ -2 & 2 & 0 & 1 \end{array} \right]$$

$$\rho_2 + \rho_1 \Rightarrow \left[ \begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 1 \end{array} \right]$$

$$\frac{1}{3}\rho_1 \Rightarrow \left[ \begin{array}{cc|cc} 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 1 & 3 & 1 & 1 \end{array} \right]$$

$$\rho_1 - \rho_2 \Rightarrow \left[ \begin{array}{cc|cc} 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{4} & \frac{3}{8} \end{array} \right]$$

$$-\frac{3}{8}\rho_2 \Rightarrow \left[ \begin{array}{cc|cc} 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{4} & \frac{3}{8} \end{array} \right]$$

$$\rho_1 - \frac{1}{3}\rho_2 \Rightarrow \left[ \begin{array}{cc|cc} 1 & 0 & \frac{1}{4} & -\frac{1}{8} \\ 0 & 1 & \frac{1}{4} & \frac{3}{8} \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{8} \\ \frac{1}{4} & \frac{3}{8} \end{bmatrix}$$

$$A_0x = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

$$A_0^{-1}A_0x = A_0^{-1}\begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

$$x = \begin{bmatrix} \frac{1}{4} & -\frac{1}{8} \\ \frac{1}{4} & \frac{3}{8} \end{bmatrix} \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{9}{8} \\ -\frac{11}{8} \end{bmatrix}$$

### Question 3

Compute the determinant of

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 2 \\ 4 & 3 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -4 & 3 \\ 1 & 1 & 1 \\ 3 & 3 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 2 \\ -1 & 3 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -5 & -10 & -15 \\ 6 & 12 & 18 \end{bmatrix}$$

- Are they Invertible?
- Which of them has full rank? Which of them has lower rank and which one?

### My Solution

#### Part 1

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 2 \\ 4 & 3 & 1 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= 0 + 16 - 6 - (0 + 6 + 4) \\ &= 0 \end{aligned}$$

This matrix is not invertible.

$$\begin{bmatrix} 2 & -4 & 3 \\ 1 & 1 & 1 \\ 3 & 3 & 1 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= 2 - 1 + 9 - 9 - 6 + 4 \\ &= -12 \end{aligned}$$

This matrix is invertible

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 2 \\ -1 & 3 & 2 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= 6 + 0 + 6 + 3 - 6 - 0 \\ &= 9 \end{aligned}$$

This matrix is invertible

$$\begin{bmatrix} 1 & 2 & 3 \\ -5 & -10 & -15 \\ 6 & 12 & 18 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= -180 - 180 - 180 + 180 + 180 + 180 \\ &= 0 \end{aligned}$$

This matrix is not invertible

## Question 4

For what value  $a$  the matrix is not invertible?

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & a & 4 \\ -3 & 1 & 2 \end{bmatrix}$$

## My Solution

A matrix is not invertible if its determinant is zero.

$$\begin{aligned} \det(A) &= 2a - 24 + 2 + 3a - 4 - 8 \\ &= 5a - 34 \end{aligned}$$

$$5a - 34 = 0$$

$$a = \frac{34}{5} \approx 6.8$$

## Question 5

Compute and apply the Householder matrix which makes transforms the first column of to a multiple of the

first one-hot vector  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  for

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 2 \\ 2 & 3 & 1 \end{bmatrix}$$

and for (hint: here subtracting is nicer)

$$A = \begin{bmatrix} 1 & -4 & 3 \\ 3 & 1 & 1 \\ \sqrt{6} & 3 & 1 \end{bmatrix}$$

## My Solution

**Question 6**

Verify that

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

Satisfies being an orthogonal matrix.

**My Solution**

$$\begin{aligned} \det(A) &= 0 + (-\sin^2(\alpha)) + 0 - \cos^2(\alpha) - 0 - 0 \\ &= -(\sin^2(\alpha) + \cos^2(\alpha)) \\ &= -1 \end{aligned}$$



## Question 7

What is the cosine of the angle between

$$(6, -6, -4, \sqrt{12}), (6, 4, 2, \sqrt{25})?$$

### My Solution

$$\begin{aligned}\cos(\angle(\vec{a}, \vec{b})) &= \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|} \\ &= \frac{6 \cdot 6 + (-6) \cdot 4 + (-4) \cdot 2 + \sqrt{12} \cdot \sqrt{25}}{\sqrt{6^2 + (-6)^2 + (-4)^2 + \sqrt{12}^2} \sqrt{6^2 + 4^2 + 2^2 + \sqrt{25}^2}} \\ &= \frac{36 - 24 - 8 + 5\sqrt{12}}{\sqrt{100}\sqrt{81}} \\ &= \frac{4}{90}\end{aligned}$$

## Question 8

Another  $3 \times 3$  affine system

- show the intermediate result when the first column is the one hot vector  $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$  for the first time
- show that the intermediate result when the matrix has row echelon form for the first time
- get the solution

$$\begin{aligned} 2x - 3y + 2z &= -4 \\ 7x + 4.5y - 1z &= 16 \\ 4x + 3y + z &= 2 \end{aligned}$$

## My Solution

$$\begin{aligned} & \left[ \begin{array}{ccc|c} 2 & -3 & 2 & -4 \\ 7 & 4.5 & -1 & 16 \\ 4 & 3 & 1 & 2 \end{array} \right] \\ \frac{1}{2}\rho_1 & \Rightarrow \left[ \begin{array}{ccc|c} 1 & -3/2 & 1 & -2 \\ 7 & 4.5 & -1 & 16 \\ 4 & 3 & 1 & 2 \end{array} \right] \\ (i)\rho_2 - 7\rho_1, \rho_3 - 4\rho_1 & \Rightarrow \left[ \begin{array}{ccc|c} 1 & -3/2 & 1 & -2 \\ 0 & 15 & -8 & 30 \\ 0 & 9 & -3 & 10 \end{array} \right] \\ \frac{1}{15}\rho_2, -\frac{1}{3}\rho_3 & \Rightarrow \left[ \begin{array}{ccc|c} 1 & -3/2 & 1 & -2 \\ 0 & 1 & -8/15 & 2 \\ 0 & -3 & 1 & -\frac{10}{3} \end{array} \right] \\ (ii)\rho_3 + 3\rho_2 & \Rightarrow \left[ \begin{array}{ccc|c} 1 & -3/2 & 1 & -2 \\ 0 & 1 & -8/15 & 2 \\ 0 & 0 & 1 & -\frac{40}{9} \end{array} \right] \\ \rho_2 + \frac{8}{15}\rho_3 & \Rightarrow \left[ \begin{array}{ccc|c} 1 & -3/2 & 1 & -2 \\ 0 & 1 & 0 & -\frac{10}{27} \\ 0 & 0 & 1 & -\frac{40}{9} \end{array} \right] \\ \rho_1 + \frac{3}{2}\rho_2 - \rho_3 & \Rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & 0 & -\frac{17}{9} \\ 0 & 1 & 0 & -\frac{10}{27} \\ 0 & 0 & 1 & -\frac{40}{9} \end{array} \right] \\ \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} -\frac{17}{9} \\ -\frac{10}{27} \\ -\frac{40}{9} \end{bmatrix} \end{aligned}$$