INF 1004 Mathematics 2
Tutorial #7

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March 4, 2023

Question 1

Compute the euclidean vector norm for vectors

$$[1, 0, 2] , [3, 4] , [-7, 2, -4, \sqrt{12}]$$

My Solution

Question 2

Compute the corresponding unit length vector for these:

$$[3, 4]\ ,\ [-1, -2, 3]\ ,\ [-7, 2, -4, \sqrt{12}]$$

My Solution

Question 3

Compute the inner product between these vectors and their angle in degrees:

$$\begin{bmatrix}3, -2, 2\end{bmatrix}, \begin{bmatrix}1, 2, 2\end{bmatrix}$$

Compute the inner product between these vectors and their angle in degrees:

$$\begin{bmatrix}1, 0, 1\end{bmatrix}, \begin{bmatrix}2, 1, -2\end{bmatrix}, \begin{bmatrix}\frac{1}{2\sqrt{2}}, -\frac{\sqrt{3}}{2}, \frac{1}{2\sqrt{2}}\end{bmatrix}$$

Sample Solutions

$$\frac{u}{\|u\|_2} \cdot \frac{v}{\|v\|_2} = \cos(\angle(u, v))$$

Part 1

$$\begin{aligned}
\langle[3, -2, 2], [1, 2, 2]\rangle &= 3 \cdot 1 + (-2) \cdot 2 + 2 \cdot 2 \\
&= 3 + (-4) + 4 \\
&= \boxed{3}
\end{aligned}$$

Angle:

Let u be $\begin{bmatrix}3, -2, 2\end{bmatrix}$ and v be $\begin{bmatrix}1, 2, 2\end{bmatrix}$

$$\begin{aligned}
\cos(\angle(u, v)) &= \frac{\langle u, v\rangle}{\|u\|_2 \cdot \|v\|_2} \\
&= \frac{\begin{bmatrix}3, -2, 2\end{bmatrix} \cdot \begin{bmatrix}1, 2, 2\end{bmatrix}}{\|\begin{bmatrix}3, -2, 2\end{bmatrix}\|_2 \cdot \|\begin{bmatrix}1, 2, 2\end{bmatrix}\|_2} \\
&= \frac{3}{\sqrt{17} \cdot \sqrt{9}} \\
&= \frac{3}{\sqrt{153}} \\
&= \frac{1}{\sqrt{17}} \\
\angle(u, v) &= \cos^{-1}(\frac{3}{\sqrt{153}}) \\
&= \boxed{75.96^\circ} \textbf{ OR} \\
&= 360^\circ - 75.96^\circ \\
&= \boxed{284.04^\circ}
\end{aligned}$$

Question 3 continued on next page. . .

Part 2

$$\begin{aligned}\langle[1,0,1],[2,1,-2]\rangle &= 1\cdot 2+0\cdot 1+1\cdot(-2)\\ &= 2+0+(-2)\\ &= \boxed{0}\end{aligned}$$

Angle:

Let u be $\begin{bmatrix}1,0,1\end{bmatrix}$ and v be $\begin{bmatrix}2,1,-2\end{bmatrix}$ and w be $\begin{bmatrix}\frac{1}{2\sqrt{2}},-\frac{\sqrt{3}}{2},\frac{1}{2\sqrt{2}}\end{bmatrix}$

$$\begin{aligned}\cos(\angle(u,v)) &= \frac{\langle u,v\rangle}{\|u\|_2\cdot\|v\|_2}\\ &= \frac{\begin{bmatrix}1,0,1\end{bmatrix}\cdot\begin{bmatrix}2,1,-2\end{bmatrix}}{\|\begin{bmatrix}1,0,1\end{bmatrix}\|_2\cdot\|\begin{bmatrix}2,1,-2\end{bmatrix}\|_2}\\ &= \frac{0}{\sqrt{2}\cdot\sqrt{6}}\\ &= 0\end{aligned}$$

$$\begin{aligned}\angle(u,v) &= \cos^{-1}(0)\\ &= \boxed{90^\circ}\ \textbf{OR}\\ &= 360^\circ-90^\circ\\ &= \boxed{270^\circ}\end{aligned}$$

Part 3

$$\left\langle [1,0,1]\cdot\left[\frac{1}{2\sqrt{2}},-\frac{\sqrt{3}}{2},\frac{1}{2\sqrt{2}}\right]\right\rangle = \frac{1}{\sqrt{2}}$$

$$\left\|\left[\frac{1}{2\sqrt{2}},-\frac{\sqrt{3}}{2},\frac{1}{2\sqrt{2}}\right]\right\| = \sqrt{\frac{1}{4\cdot 2}+\frac{3}{4}+\frac{1}{4\cdot 2}} = 1$$

$$\begin{aligned}\cos\angle(u,w) &= \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{1}\\ &= \frac{1}{2}\\ \angle(u,w) &= \cos^{-1}(\frac{1}{2})\\ &= 60^\circ\ \textbf{OR}\\ \angle(u,w) &= 360^\circ-60^\circ = 300^\circ\end{aligned}$$

Part 4

$$\begin{aligned}
[2,1,-2]\cdot\left[\frac{1}{2\sqrt{2}},-\frac{\sqrt{3}}{2},\frac{1}{2\sqrt{2}}\right] &= \frac{1}{2\sqrt{2}}\cdot 2+\frac{\sqrt{3}}{2}\cdot 1+\frac{1}{2\sqrt{2}}\cdot(-2)\\
&= -\frac{\sqrt{3}}{2}
\end{aligned}$$

$$\begin{aligned}
\cos\angle(v,w) &= -\frac{\sqrt{3}}{2}\cdot\frac{1}{3}\cdot\frac{1}{1}\\
&= -\frac{1}{2\sqrt{3}}\\
\angle(v,w) &= \cos^{-1}(-\frac{1}{2\sqrt{3}})\\
&= 106.74^\circ\ \textbf{OR}\\
\angle(v,w) &= 360^\circ - 106.74^\circ = 253.26^\circ
\end{aligned}$$

Question 4

- What is the projection of $\begin{bmatrix}5, 2\end{bmatrix}$ onto the subspace spanned by vector $\begin{bmatrix}1, 1\end{bmatrix}$?
- What is the projection of $\begin{bmatrix}0, 2, 1\end{bmatrix}$ onto the subspace spanned by vector $\begin{bmatrix}1, -1, -1\end{bmatrix}$?
- Project $\begin{bmatrix}5, 2\end{bmatrix}$ onto the subspace spanned by vectors $\begin{bmatrix}2, 3\end{bmatrix}$, $\begin{bmatrix}1, 1\end{bmatrix}$
- What is the projection of $\begin{bmatrix}1, -1, 1\end{bmatrix}$ onto the subspace spanned by vector $\begin{bmatrix}1, 1, 1\end{bmatrix}$ onto the subspace spanned by vectors $\begin{bmatrix}0, 0, -1\end{bmatrix}$, $\begin{bmatrix}2, 0, 1\end{bmatrix}$? Hint: this one is more tricky. Reason: $\begin{bmatrix}0, -1, -1\end{bmatrix} \cdot \begin{bmatrix}2, 0, 1\end{bmatrix} \neq 0$

My Solution

Question 5

Compute these matrix multiplications:

$$\begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ -4 & -2 \end{bmatrix}$$

$$\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \end{bmatrix}$$

Question: Do you need more of them to practice? If so, you can do at home:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 0.5 & 2.5 \\ -3.5 & 1.5 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

My Solution

Question 6

- Project $\begin{bmatrix}5, 2\end{bmatrix}$ onto the orthogonal space of vector $\begin{bmatrix}2, -3\end{bmatrix}$
- Project $\begin{bmatrix}1, -1, 3\end{bmatrix}$ onto the orthogonal space of vector $\begin{bmatrix}-3, 1, 1\end{bmatrix}$
- Project $\begin{bmatrix}1, -1, 3, 1\end{bmatrix}$ onto the orthogonal space of vector $\begin{bmatrix}-2, 2, 0, 0\end{bmatrix}$, $\begin{bmatrix}0, 0, \sqrt{2}, \sqrt{2}\end{bmatrix}$

My Solution

Question 7

Run Gram-Schmid-orthogonalization on the vectors

$$\left[12, 12, 6\right], \left[2, -2, 4\right], \left[-2, -2, 1\right]$$

My Solution

Question 8

Understanding Distances coming from ℓ_p-norms

Coding: plot in python or similar the set of points $x \in \mathbb{R}^2$ usuch that $\|x\|_p = 1$ for

- $p = 0.2$
- $p = 0.5$
- $p = 1$
- $p = 1.5$
- $p = 2$
- $p = 4$
- $p = 8$
- $p = 16$

Hint: in 2 dimensions for $p = 2$ the solution is given by

$$x(t) = (\cos(t), \sin(t))$$

due to $\cos^2(t) + \sin^2(t) = 1$.
you can use the same idea with different powers. You can start by considering $(\cos^r(t), \sin^r(t))$. One thing to node: $\cos(t)^r + \sin(t)^r$ is not always defined for negative values and certain r.

For $p \neq 2$, you can consider this, thich deals with the signs:

$$x(t) = (sign(\cos(t))|\cos(t)|^r, sign(\sin(t))|\sin(t)|^r)$$

for the right choice of r. Find out which r is suitable for a general $p > 0$ such that $\|x\|_p = 1$. Then plot it in python.

My Solution