INF 1004 Mathematics 2
Tutorial #1 & #2 Solutions

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Question 1

In a sample of 20 people the times taken, in seconds, to solve a simple numerical puzzle were as follows:

17	19	22	26	28	31	34	36	38	39
41	42	43	47	50	51	53	55	57	58

1. Calculate the mean and standard deviation of these times.
2. In fact, 23 people attempted to solve this puzzle. However, 3 of them failed to solve it within the allotted time of 60 seconds, taking 62, 65 and 97 seconds respectively. Calculate the interquartile range (IQR) of the times taken by all 23 people.
3. Are there any outliers in the dataset?

My Solution
Part 1

Mean

$$
\begin{aligned}
\bar{x} &= \frac{1}{n}\sum_{i=1}^{n} x_i \\
&= \frac{(17+19+22+26+28+31+34+36+38+39+41+42+43+47+50+51+53+55+57+58)}{20} \\
&= \frac{787}{20} \\
&= 39.35
\end{aligned}
$$

Standard Deviation

$$
\begin{aligned}
s^2 = \frac{\sum(x_i-\bar{x})^2}{n-1} &= \frac{1}{n-1}\left[\sum x_i^2 - \frac{(\sum x_i)^2}{n}\right] \\
&= \frac{1}{19}\left[\sum x_i^2 - \frac{(\sum x_i)^2}{20}\right] \\
&= \frac{1}{19}\left[\sum x_i^2 - \frac{787^2}{20}\right] \\
&= \frac{1}{19}\left[(17^2+19^2+\ldots+58^2) - \frac{787^2}{20}\right] \\
&= 12.86
\end{aligned}
$$

The mean is $\boxed{39.35}$ and the standard deviation is $\boxed{12.68}$.

Question 1 continued on next page. . .

Part 2

Interquartile Range

Rearranging the data in ascending order, we get:

17, 19, 22, 26, 28, 31, 34, 36, 38, 39, 41, 42, 43, 47, 50, 51, 53, 55, 57, 58, 62, 65, 97.

Hence, Q1 = 31 and Q3 = 55.

$$\begin{aligned} \mathrm{IQR} &= \mathrm{Q3} - \mathrm{Q1} \\ &= 55 - 31 \\ &= 24 \end{aligned}$$

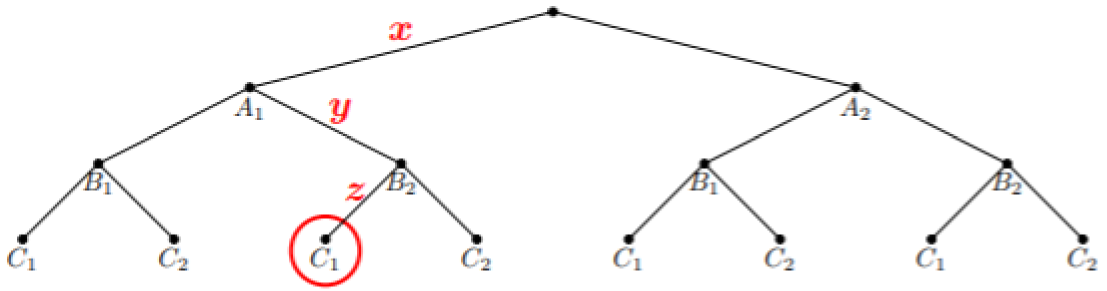
Part 3

Outliers

The interquartile range is 24.
$1.5 \times \mathrm{IQR} = 36$.
$Q3 + 1.5 \times \mathrm{IQR} = 55 + 36 = 91$.
$Q1 - 1.5 \times \mathrm{IQR} = 31 - 36 = -5$.
Hence, the outlier is 97, since it is greater than $1.5 \times \mathrm{IQR}$

Question 2

Consider the below tree diagram.



i. The probability x represents (MCQ)?

(a) $P(A_1)$

(b) $P(A_1|B_2)$

(c) $P(B_2|A_1)$

(d) $P(C_1|B_2 \cap A_1)$

ii. The probability y represents (MCQ)?

(a) $P(B_2)$

(b) $P(A_1|B_2)$

(c) $P(B_2|A_1)$

(d) $P(C_1|B_2 \cap A_1)$

iii. The probability z represents (MCQ)?

(a) $P(C_1)$

(b) $P(B_2|C_1)$

(c) $P(C_1|B_2)$

(d) $P(C_1|B_2 \cap A_1)$

iv. The circled node represents the event (MCQ)?

(a) C_1

(b) $B_2 \cap C_1$

(c) $A_1 \cap B_2 \cap C_1$

(d) $C_1|B_2 \cap A_1$

[v. and vi. do not refer to the tree diagram above]

v. Let A and B be two events. Suppose that the probability that neither event occurs is 3/8. What is the probability that at least one of the events occurs?

vi. Let C and D be two events. Suppose P(C) = 0.5, $P(C \cap D) = 0.2$ and $P((C \cup D)\ complement) = 0.4$. What is P(D)?

Question 2 continued on next page. . .

My Solution

i. The probability x represents (MCQ)? $x = P(A_1)$

ii. The probability y represents (MCQ)? $y = P(B_2|A_1)$

iii. The probability z represents (MCQ)? $z = P(C_1|B_2 \cap A_1)$

iv. The circled node represents the event (MCQ)? $A_1 \cap B_2 \cap C_1$

v. Since $P(A^c \cap B^c) = \frac{3}{8}$ and $P(A) + P(B) + P(A \cap B) = 1$, $P(\text{At least 1 event occurs}) = 1 - \frac{3}{8} = \frac{5}{8}$

Given:
$P(C) = 0.5$
$P(C \cap D) = 0.2$
$P((C \cup D)^c) = 0.4$
$P(C \cup D) = 1 - 0.4 = 0.6$

$P(C \cup D) = P(C) + P(D) - P(C \cap D)$
$\rightarrow 0.6 = 0.5 + P(D) - 0.2$
$\rightarrow P(D) = 0.3$

Question 3

In a survey of 1929 students, the following data were obtained on "students' first reason for application to the university in which they matriculated."

Enrollment status	Reason for Application: University Quality	University Cost or Convenience	Other	Total
Full-time	421	393	76	890
Part-time	400	593	46	1039
Total	821	986	122	1929

(a) What is the probability that university quality is the first reason for a student to choose a university?

(b) For a full-time student, what is the probability that university quality is the first reason for choosing a university?

(c) Let A denote the event that a student is full-time, and let B denote the event that the student lists university quality as the first reason for applying. Are events A and B independent?

My Solution
Part 1

$P(\text{quality}) = \frac{821}{1929} \approx 0.426$

Part 2

$P(\text{quality}|\text{full-time}) = \frac{P(\text{quality}\cap\text{full-time})}{P(\text{full-time})} = \frac{\frac{421}{1929}}{\frac{890}{1929}} \approx 0.473$

Part 3

A: A student is full-time
B: the student lists university quality as the first reason for applying.
Are events A and B independent?
First, check if $P(B) = P(B|A)$

$$P(B) = P(\text{University Quality}) = \frac{821}{1929} = 0.426$$
$$P(B|A) = P(\text{University Quality}|\text{full time student}) = \frac{421}{890} = 0.473$$

Since $P(B|!A) \neq P(B)$, the events are not independent.

Question 4

A rare blood disease is found in 2% of a certain population. A new blood test can correctly identify 96% of the people with the disease and 94% of the people without the disease.

(a) What is the probability that a person who is tested positive by the blood test actually has the disease?

(b) What is the probability that a person who is tested negative by the blood test actually does not have the disease?

My Solution

Let D = the person has the disease
Let T = the person tests positive

$P(D) = 0.02$
$P(T|D) = 0.96$
$P(T^c|D^c) = 0.94$

Part 1

$$\begin{aligned} P(D|T) &= \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|D^c)P(D^c)} \\ &= \frac{0.96 \times 0.02}{0.96 \times 0.02 + 0.06 \times 0.98} \\ &= 0.246 \end{aligned}$$

Part 2

$$\begin{aligned} P(D^c|T^c) &= \frac{P(T^c|D^c)P(D^c)}{P(T^c|D^c)P(D^c) + P(T^c|D)P(D)} \\ &= \frac{0.94 \times 0.98}{0.94 \times 0.98 + 0.04 \times 0.02} \\ &= 0.999 \end{aligned}$$

Question 5

A woman is pregnant with male twins. Twins may be either identical or fraternal (non- identical). In general, $\frac{1}{3}$ twins born are identical. Obviously, identical twins must be of the same sex; fraternal twins may or may not be. Assume that identical twins are equally likely to be both boys or both girls, while for fraternal twins all possibilities are equally likely: boy-girl, girl-boy, boy-boy, girl-girl. Given the above information, what is the probability that the woman's male twins are identical?

My Solution

$$P(\text{identical}|BB) = \frac{P(BB|\text{identical})P(\text{identical})}{P(BB)} = \frac{\frac{1}{2}\cdot\frac{1}{3}}{\frac{1}{2}\cdot\frac{1}{3}+\frac{1}{4}\cdot\frac{2}{3}} = \frac{1}{2}$$

Question 6

If a family had three kids, named Alice, Bob, and Carl. Assume that each is equally likely to be born; I.E., $\frac{1}{3}$ chance for each of them to be born first etc.

(a) Find the probability that Alice is older than Bob, given that Alice is older than Carl.

(b) Is event “Alice is older than Bob” independent from event “Alice is older than C”?

My Solution

A)
Let A represent the event of Alice born
Let B represent the event of Bob born
Let C represent the event of Carl born

$$\begin{aligned} P(A > B|A > C) &= \frac{P(A > B \cap A > C)}{P(A > C)} \\ &= \frac{\frac{1}{3}}{\frac{1}{2}} \\ &= \frac{2}{3} \end{aligned}$$

B)
They are not independent because $P(A > B|A > C) \neq P(A > B)$

Question 7

A boy receives a school report card each week. He is given a special treat whenever his report indicates "Good behavior" **AND** "Excellent Homework". His behavior is good with probability 0.6. When his behavior is good, he has a probability of 0.8 of doing excellent homework. When his behavior is not good, his probability of doing excellent homework is only 0.5.

a) For a random week, calculate the probability that he will be given a special treat

b) For a random Week, calculate the probability that his homework will be excellent but he will not be given a special treat

c) For a random Week, calculate the probability that his homework will be excellent

d) Given that his homework is excellent, calculate the conditional probability that he is NOT given a special treat.

My Solution

Let G represent the event of good behavior
Let E represent the event of excellent homework
E^c is the complement of E and G^c is the complement of G

A)
"Good Behavior" and "Excellent Homework"

$$P(G \cap E) = P(G)P(E|G) = 0.6 \times 0.8 = 0.48$$

B)
"Not Good behavior" AND "Excellent Homework"

$$P(G^c \cap E) = P(G^c)P(E|G^c) = 0.4 \times 0.5 = 0.2$$

C)
"Excellent Homework"

$$P(E) = P(G)P(E|G) + P(G^c)P(E|G^c) = 0.6 \times 0.8 + 0.4 \times 0.5 = 0.68$$

D)
"Not Good Behavior" given "Excellent Homework"

$$P(G^c|E) = \frac{P(G^c \cap E)}{P(E)} = \frac{0.2}{0.68} = 0.294$$

Question 8

In an attempt to find the mean number of hours his tutorial classmates spent per day preparing for tutorials, John collected data from 10 of his friends in the tutorial group and found that the sample mean is 2.4 hours with a sample standard deviation of 0.8 hours. However, a day later he felt that the sample size is too small. So he collected data from another 5 of his friends and found that the sample mean is 2.0 hours with a sample standard deviation of 1.2 hours.
Find the sample mean and sample standard deviation <u>when these 2 sets of data are combined</u>.

My Solution

Given:

$$X = \{x_1, x_2, \ldots, x_{10}\}$$
$$Y = \{y_1, y_2, \ldots, y_5\}$$

$$\begin{aligned}
\bar{x} &= \frac{\sum x_i}{10} = 2.4 \\
\sum x_i &= 10 \times 2.4 = 24 \\
\bar{y} &= \frac{\sum y_i}{5} = 2.0 \\
\sum y_i &= 5 \times 2.0 = 10 \\
\text{new mean } \bar{z} &= \frac{\sum x_i + \sum y_i}{n_x + n_y} = \frac{24 + 10}{10 + 5} = 2.267
\end{aligned}$$

Sample Standard Deviation can be used to figure out the missing values

$$\begin{aligned}
S_x^2 &= \frac{1}{10-1}\left[\sum x_i^2 - \frac{(\sum x_i)^2}{10}\right] = 0.8^2, \text{ so } \sum x_i^2 = 9 \times 0.8^2 + \frac{24^2}{10} = 63.36 \\
S_y^2 &= \frac{1}{5-1}\left[\sum y_i^2 - \frac{(\sum y_i)^2}{5}\right] = 1.2^2, \text{ so } \sum y_i^2 = 4 \times 1.2^2 + \frac{10^2}{5} = 25.76 \\
\text{new variance } s_z^2 &= \frac{1}{10+5-1}\left[\sum x_i^2 + \sum y_i^2 - \frac{(\sum x_i + \sum y_i)^2}{10+5}\right] \\
&= \frac{1}{14}\left[63.36 + 25.76 - \frac{(24+10)^2}{15}\right] = 0.86 \\
\text{new Standard Deviation } s_z &= \sqrt{s_z^2} = \sqrt{0.86} = 0.927
\end{aligned}$$

Question 9

A certain family has 6 children, consisting of 3 boys and 3 girls. Assuming that all birth orders are equally likely, what is the probability that the 3 eldest children are the 3 girls?

My Solution

$$\begin{aligned}\text{Probability that the 3 eldest are the 3 girls} &= \frac{3}{6} \times \frac{2}{5} \times \frac{1}{4} \\ &= \frac{6}{120} \\ &= 0.05\end{aligned}$$

Note: Only one possible way the 3 girls can be the oldest