INF1004 Some observed mistakes

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Removing a vector from another

When it is written to remove vector v from vector u, mind the order of which one to subtract from which one:
To remove v **from** u means to compute a vector $u - cv$ which is orthogonal to v:

$$u - cv \text{ such that}$$
$$(u - cv) \cdot v = 0$$

How is the real number c to be determined ?

$$c = \frac{v \cdot u}{v \cdot v}$$
$$\text{compute: } u - \frac{v \cdot u}{v \cdot v} v$$

If you do not wish to remember the formula, then derive it:

$$(u - cv) \cdot v = 0 \Leftrightarrow u \cdot v - cv \cdot v = 0 \Leftrightarrow u \cdot v = cv \cdot v$$
$$\Leftrightarrow c = \frac{u \cdot v}{v \cdot v}$$

The cosine of the angle between two vectors u, v is

$$\frac{u \cdot v}{\|u\|_2 \|v\|_2}$$

If it is said that the cosine of the angle is for example $\frac{3}{8}$, then it means

$$\frac{u \cdot v}{\|u\|_2 \|v\|_2} = \frac{3}{8}$$

It does **not** mean $\cos(\frac{3}{8})$.
Cosine of the angle is **not** the angle, too. Lots of students did that wrong. This is a matter of reading.

Many students have difficulties solving 3×3 affine systems, with or without calculator.
Here my suggestions. However, practice to find out wwidehat fits your brain best.

- train using 3×3 which you write down yourself. To do this:
- Write down a system with integers, and multiples of $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ like this:

$$\left(\begin{array}{ccc|c} -3 & 2 & -2 & 5 \\ 7 & -4.5 & 8 & 13 \\ 2 & 6 & 3.5 & -7 \end{array}\right)$$

- compute its solution using *numpy.linalg.solve*. This works only if $det(A) = 0$. You can tweak it until you like its solution.

Solving affine equation systems II

- if you want to solve systems with a vector space of solutions, start off a matrix in a nice shape like

$$\left(\begin{array}{ccc|c} 1 & -6 & -0 & 5 \\ 0 & 3 & 5 & 13 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

  then

  - add multiples of the first row to the second row
  - add multiples of zero-th row to the first and second row

  ... until you have a starting point for solving it

- You can do the same also for the case above to construct "nice" solutions

Solving affine equation systems III

Here a suggestion when solving in an examination:

- If you add a multiple row to another row, like $R_1 = R_1 - 5R_0$, then consider whether the following would assist you:
  - write every operation on paper first, then compute it using the calculator.
  - For example: $R_1 = R_1 - 5R_0$ in

$$\left(\begin{array}{ccc|c} 1 & -3 & -5 & 13 \\ 5 & 2 & 4 & 11 \\ 7 & 3 & 1 & -3 \end{array}\right)$$

then write down
$-5 * -3 + 2$,
$-5 * -5 + 4$,
$-5 * 13 + 11$
Reason: to reduce mistakes wwidehat you are typing into calculators

Solving affine equation systems III

- For a quick check, compute the determinant of A.
  - if $det(A) = 0$, then you know that you have either a vector space of solutions or no solutions. $det(A) = 0$ implies for a non-zero 3×3 matrix that you end up in a case like

$$\left(\begin{array}{ccc|c} 1 & * & * & b_0 \\ 0 & 1 & * & b_1 \\ 0 & 0 & 0 & b_2 \end{array}\right) \; or \; \left(\begin{array}{ccc|c} 1 & * & * & b_0 \\ 0 & 0 & 0 & b_1 \\ 0 & 0 & 0 & b_2 \end{array}\right)$$

  - Depending on the values of b_1, b_2 this may have a vector space or no solution.
  - If $det(A) = 0$, any correct transformation of A must result in a matrix $\widehat{A}$ with $det(\widehat{A}) = 0$. It cannot become non-zero in such a case.

Getting a vector space solution I

If you have this,

$$\left(\begin{array}{ccc|c} 1 & * & * & b_0 \\ 0 & 1 & * & b_1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

then the proper way is to at first put it into reduced row echelon form:

$$\left(\begin{array}{ccc|c} 1 & 0 & a_{02} & b_0 \\ 0 & 1 & a_{12} & b_1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

You have in this case a 1-dim dependency! From here the solution is (with variables x_0, x_1, x_2):

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ 0 \end{pmatrix} + s \begin{pmatrix} -a_{02} \\ -a_{12} \\ 1 \end{pmatrix}$$

Getting a vector space solution II

If you have this, the result is analogous, but with 2 degrees of freedom

$$\left(\begin{array}{cccc|c} 1 & \color{red}{0} & * & * & b_0 \\ 0 & 1 & * & * & b_1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \text{ or } \left(\begin{array}{ccc|c} 1 & * & * & b_0 \\ 0 & 0 & 0 & b_1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Getting a vector space solution III

$$\begin{pmatrix} 1 & {\color{red}0} & a_{02} & a_{03} & \vline & b_0 \\ 0 & 1 & a_{12} & a_{13} & \vline & b_1 \\ 0 & 0 & 0 & 0 & \vline & 0 \end{pmatrix} \to \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -a_{02} \\ -a_{12} \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -a_{03} \\ -a_{13} \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & a_{02} & a_{03} & \vline & b_0 \\ 0 & 0 & 0 & \vline & b_1 \\ 0 & 0 & 0 & \vline & 0 \end{pmatrix} \to \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ 0 \end{pmatrix} + s \begin{pmatrix} -a_{02} \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -a_{03} \\ 0 \\ 1 \end{pmatrix}$$

Getting a vector space solution III

$$\begin{pmatrix} 1 & \color{red}{0} & \color{red}{0} & a_{03} & \vert & b_0 \\ 0 & 1 & \color{red}{0} & a_{13} & \vert & b_1 \\ 0 & 0 & 1 & a_{23} & \vert & b_2 \\ 0 & 0 & 0 & 0 & \vert & 0 \end{pmatrix}$$

would result in a 1-dimensional degree of freedom solution for

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

- if $det(A) \neq 0$, then you know that the solution of $Ax = b$ will be a single vector x.
- you can use $det(A)$ for a partial check of your transformations on the A part of the extended matrix $A|b$:
  - if $\widehat{A}$ is obtained from A by **multiplying a single column** with a constant c, then $det(\widehat{A}) = c * det(A)$
  - if $\widehat{A}$ is obtained from A by adding a multiple of one column to another $(R_1 = R_1 - 5R_0)$, then $det(\widehat{A}) = det(A)$
- this allows you in the case of $det(A) \neq 0$ to identify whether a mistake happened somewhere in your transformations (when the determinant changes in a different way from the above).
- This is not a foolproof guarantee to find a mistake. Under certain conditions, the determinant might not change, if you do a wrong transformation

  Example:

$$det(\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}) = -bc$$

  does not depend on the value of a. Also this does not help to find mistakes in transforming the bias vector b.