INF 1004 Mathematics 2
Tutorial #10 Solutions

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Question 1

Compute $A^\top A$ for

$$A = \begin{bmatrix} 2 & -3 \\ -6 & -9 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 0 & 2 \\ 4 & 3 & 1 \end{bmatrix}$$

My Solution

Part 1

$$\begin{aligned} A &= \begin{bmatrix} 2 & -3 \\ -6 & -9 \end{bmatrix} \\ A^\top &= \begin{bmatrix} 2 & -6 \\ -3 & -9 \end{bmatrix} \\ A^\top A &= \begin{bmatrix} 2 & -6 \\ -3 & -9 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -6 & -9 \end{bmatrix} \\ &= \begin{bmatrix} 40 & 48 \\ 48 & 100 \end{bmatrix} \end{aligned}$$

Part 2

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 & -1 \\ 2 & 0 & 2 \\ 4 & 3 & 1 \end{bmatrix} \\ A^\top &= \begin{bmatrix} 0 & 2 & 4 \\ 1 & 0 & 3 \\ -1 & 2 & 1 \end{bmatrix} \\ A^\top A &= \begin{bmatrix} 0 & 2 & 4 \\ 1 & 0 & 3 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ 2 & 0 & 2 \\ 4 & 3 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 20 & 12 & 8 \\ 12 & 10 & 2 \\ 8 & 2 & 6 \end{bmatrix} \end{aligned}$$

Question 2

Compute the inverse of

$$A_0 = \begin{bmatrix} -2 & -3 \\ -6 & -4 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 3 & 1 \\ -2 & 2 \end{bmatrix}$$

Use this inverses to solve

$$A_0 x = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$A_0 x = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

Note: it is not common to solve $Ax = b$ using matrix inversion.

Reasons:

- $Ax = b$ can be solvable when A is not invertible.
- It is often slower / more costly

My Solution

Part 1

$$\begin{aligned}
A^{-1} &\implies \left[\begin{array}{cc|cc} -2 & -3 & 1 & 0 \\ -6 & -4 & 0 & 1 \end{array}\right] \\
-\frac{1}{2}\rho_1, -\frac{1}{4}\rho_2 &\implies \left[\begin{array}{cc|cc} 1 & \frac{3}{2} & -\frac{1}{2} & 0 \\ \frac{3}{2} & 1 & 0 & -\frac{1}{4} \end{array}\right] \\
\frac{3}{2}\rho_1 - \rho_2 &\implies \left[\begin{array}{cc|cc} 1 & \frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{5}{4} & -\frac{3}{4} & \frac{1}{4} \end{array}\right] \\
\frac{4}{5}\rho_2 &\implies \left[\begin{array}{cc|cc} 1 & \frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{3}{5} & \frac{1}{5} \end{array}\right] \\
\rho_1 - \frac{3}{2}\rho_2 &\implies \left[\begin{array}{cc|cc} 1 & 0 & \frac{2}{5} & -\frac{3}{10} \\ 0 & 1 & -\frac{3}{5} & \frac{1}{5} \end{array}\right]
\end{aligned}$$

$$A^{-1} = \begin{bmatrix} \frac{2}{5} & -\frac{3}{10} \\ \frac{3}{5} & \frac{1}{10} \end{bmatrix}$$

$$\begin{aligned}
A_0 x &= \begin{bmatrix} 3 \\ 1 \end{bmatrix} \\
A_0^{-1} A_0 x &= A_0^{-1} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \\
x &= \begin{bmatrix} \frac{2}{5} \\ \frac{3}{5} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 0.9 \\ -1.6 \end{bmatrix}
\end{aligned}$$

Question 2 continued on next page...

Part 2

$$
\begin{aligned}
A^{-1} &\implies \left[\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ -2 & 2 & 0 & 1 \end{array}\right] \\
\rho_2 + \rho_1 &\implies \left[\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 1 \end{array}\right] \\
\frac{1}{3}\rho_1 &\implies \left[\begin{array}{cc|cc} 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 1 & 3 & 1 & 1 \end{array}\right] \\
\rho_1 - \rho_2 &\implies \left[\begin{array}{cc|cc} 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{4} & \frac{3}{8} \end{array}\right] \\
-\frac{3}{8}\rho_2 &\implies \left[\begin{array}{cc|cc} 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{4} & \frac{3}{8} \end{array}\right] \\
\rho_1 - \frac{1}{3}\rho_2 &\implies \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{4} & -\frac{1}{8} \\ 0 & 1 & \frac{1}{4} & \frac{3}{8} \end{array}\right] \\
A^{-1} &= \begin{bmatrix} \frac{1}{4} & -\frac{1}{8} \\ \frac{1}{4} & \frac{3}{8} \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
A_0 x &= \begin{bmatrix} 2 \\ -5 \end{bmatrix} \\
A_0^{-1} A_0 x &= A_0^{-1} \begin{bmatrix} 2 \\ -5 \end{bmatrix} \\
x &= \begin{bmatrix} \frac{1}{4} & -\frac{1}{8} \\ \frac{1}{4} & \frac{3}{8} \end{bmatrix} \begin{bmatrix} 2 \\ -5 \end{bmatrix} \\
&= \begin{bmatrix} \frac{9}{8} \\ -\frac{11}{8} \end{bmatrix}
\end{aligned}
$$

Sample Solutions

A:

$$
\begin{aligned}
\det(A_0) &= (-2)(-4) - (-3)(-6) = -10 \\
A_0^{-1} &= \frac{1}{\det(A_0)} \begin{bmatrix} -4 & 3 \\ 6 & -2 \end{bmatrix} \\
x &= A_0^{-1} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \\
&= -\frac{1}{10} \begin{bmatrix} -4 & 3 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 0.9 \\ -1.6 \end{bmatrix}
\end{aligned}
$$

B:

$$
\begin{aligned}
\det A_1 &= 3(2) - (-2)(1) = 8 \\
A_1^{-1} &= \frac{1}{8} \begin{bmatrix} 2 & -1 \\ 2 & 3 \end{bmatrix} \\
x &= \frac{1}{8} \begin{bmatrix} 2 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -5 \end{bmatrix} \\
&= \begin{bmatrix} \frac{9}{8} \\ -\frac{11}{8} \end{bmatrix}
\end{aligned}
$$

FORMULA:

$$
A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}
$$

Question 3

Compute the determinant of

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 2 \\ 4 & 3 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -4 & 3 \\ 1 & 1 & 1 \\ 3 & 3 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 2 \\ -1 & 3 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -5 & -10 & -15 \\ 6 & 12 & 18 \end{bmatrix}$$

- Are they Invertible?
- Which of them has full rank? Which of them has lower rank and which one?

Notes:

- Rank is the number of linearly independent rows or columns.
- A matrix is invertible if they have full rank.

Sample Solutions

Part 1

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 2 \\ 4 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= 0 + 16 - 6 - (0 + 6 + 4) \\ &= 0 \end{aligned}$$

This matrix is not invertible. Rank is 2.

Part 2

$$\begin{bmatrix} 2 & -4 & 3 \\ 1 & 1 & 1 \\ 3 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -\frac{5}{6} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= 2 - 1 + 9 - 9 - 6 + 4 \\ &= -12 \end{aligned}$$

This matrix is invertible. Rank is 3.

Part 3

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 2 \\ -1 & 3 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= 6 + 0 + 6 + 3 - 6 - 0 \\ &= 9 \end{aligned}$$

This matrix is invertible. Rank is 3.

Part 4

$$\begin{bmatrix} 1 & 2 & 3 \\ -5 & -10 & -15 \\ 6 & 12 & 18 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= -180 - 180 - 180 + 180 + 180 + 180 \\ &= 0 \end{aligned}$$

This matrix is not invertible and has rank 1.

Question 4

For what value a the matrix is not invertible?

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & a & 4 \\ -3 & 1 & 2 \end{bmatrix}$$

My Solution

A matrix is not invertible if its determinant is zero.

$$\begin{aligned} \det(A) &= 2a - 24 + 2 + 3a - 4 - 8 \\ &= 5a - 34 \\ 5a - 34 &= 0 \\ a &= \frac{34}{5} \approx 6.8 \end{aligned}$$

Question 5

Compute and apply the Householder matrix which makes transforms the first column of to a multple of the first one-hot vector $\begin{bmatrix}1\\0\\0\end{bmatrix}$ for

$$A = \begin{bmatrix}1 & 2 & -1\\2 & 0 & 2\\2 & 3 & 1\end{bmatrix}$$

and for (hint: here subtracting is nicer)

$$A = \begin{bmatrix}1 & -4 & 3\\3 & 1 & 1\\\sqrt{6} & 3 & 1\end{bmatrix}$$

Formula:

$$u = x \pm \|x\|_2 z$$
$$H_u = I - \frac{2}{u \cdot u} uu^T$$

Properties of H:

$$H_u x = \mp \|x\|_2 z \text{ H maps x to } \mp |x| z \text{ given } u = x \pm |x| z$$
$$H_u^\top H_u = I \text{ H is an orthogonal matrix}$$
$$H_u = H_u^\top \text{ H is a symmetric matrix}$$

Example:

Here $A = \begin{bmatrix}3 & 1\\4 & 4\end{bmatrix}$

$x = \begin{bmatrix}3\\4\end{bmatrix}$

$x_2 = \sqrt{3^2+4^2} = \sqrt{25} = 5$

$$\begin{aligned}
u &= \begin{bmatrix}3\\4\end{bmatrix} \pm 5 \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}-2\\4\end{bmatrix} \text{ (Choose Minus)}\\
H &= \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \frac{2}{(-2)^2+4^2}\begin{bmatrix}-2\\4\end{bmatrix}\begin{bmatrix}-2 & 4\end{bmatrix}\\
&= \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \frac{2}{20}\begin{bmatrix}4 & -8\\-8 & 16\end{bmatrix}\\
&= \begin{bmatrix}0.6 & 0.8\\0.8 & -0.6\end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
HA &= \begin{bmatrix}0.6 & 0.8\\0.8 & -0.6\end{bmatrix}\begin{bmatrix}3 & 1\\4 & 4\end{bmatrix}\\
&= \begin{bmatrix}5 & 3.8\\0 & -1.6\end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
u &= \begin{bmatrix}3\\4\end{bmatrix} \pm 5 \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}8\\4\end{bmatrix} \text{ (Choose Positive Sign)}\\
H &= \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \frac{2}{(8)^2+4^2}\begin{bmatrix}8\\4\end{bmatrix}\begin{bmatrix}8 & 4\end{bmatrix}\\
&= \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \frac{2}{80}\begin{bmatrix}64 & 32\\32 & 16\end{bmatrix}\\
&= \begin{bmatrix}-0.6 & -0.8\\-0.8 & 0.6\end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
HA &= \begin{bmatrix}-0.6 & -0.8\\-0.8 & 0.6\end{bmatrix}\begin{bmatrix}3 & 1\\4 & 4\end{bmatrix}\\
&= \begin{bmatrix}-5 & -3.8\\0 & -1.6\end{bmatrix}
\end{aligned}$$

Question 5 continued on next page. . .

Part 1

$$
\begin{aligned}
u &= \begin{bmatrix}1\\2\\2\end{bmatrix} \pm \|(1,2,2)\| \begin{bmatrix}1\\0\\0\end{bmatrix} \\
&= \begin{bmatrix}1\\2\\2\end{bmatrix} \pm \sqrt{1^2+2^2+2^2} \begin{bmatrix}1\\0\\0\end{bmatrix} \\
&= \begin{bmatrix}1\\2\\2\end{bmatrix} - 3 \begin{bmatrix}1\\0\\0\end{bmatrix} \\
&= \begin{bmatrix}1\\2\\2\end{bmatrix} - \begin{bmatrix}3\\0\\0\end{bmatrix} \\
&= \begin{bmatrix}-2\\2\\2\end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
H &= I - \frac{2}{12}uu^\top = I - \frac{1}{6}\begin{bmatrix}-2\\2\\2\end{bmatrix}\begin{bmatrix}-2 & 2 & 2\end{bmatrix} \\
&= I - \frac{1}{6}\begin{bmatrix}4 & -4 & -4\\-4 & 4 & 4\\-4 & 4 & 4\end{bmatrix} \\
&= \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} - \frac{1}{6}\begin{bmatrix}4 & -4 & -4\\-4 & 4 & 4\\-4 & 4 & 4\end{bmatrix} \\
&= \begin{bmatrix}\frac{1}{3} & \frac{2}{3} & \frac{2}{3}\\ \frac{2}{3} & \frac{1}{3} & \frac{1}{3}\\ \frac{2}{3} & \frac{1}{3} & \frac{1}{3}\end{bmatrix} \\
HA &= \begin{bmatrix}3 & 2.67 & 1.67\\0 & -0.67 & -0.67\\0 & 2.33 & -1.67\end{bmatrix}
\end{aligned}
$$

Part 2

$$
\begin{aligned}
u &= x - \|x\|z = \begin{bmatrix}1\\3\\\sqrt{6}\end{bmatrix} - 4\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}-3\\3\\\sqrt{6}\end{bmatrix} \\
H &= I - \frac{2}{u\cdot u}uu^\top \\
&= I - \frac{2}{24}\begin{bmatrix}-3\\3\\\sqrt{6}\end{bmatrix}\begin{bmatrix}-3 & 3 & \sqrt{6}\end{bmatrix} \\
&= \begin{bmatrix}\frac{1}{4} & \frac{3}{4} & \frac{\sqrt{6}}{4}\\ \frac{3}{4} & \frac{1}{4} & -\frac{\sqrt{6}}{4}\\ \frac{\sqrt{6}}{4} & -\frac{\sqrt{6}}{4} & \frac{1}{2}\end{bmatrix} \\
HA &= \begin{bmatrix}4 & * & *\\0 & * & *\\0 & * & *\end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
u &= x + \|x\|z = \begin{bmatrix}1\\3\\\sqrt{6}\end{bmatrix} + 4\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}5\\3\\\sqrt{6}\end{bmatrix} \\
H &= I - \frac{2}{u\cdot u}uu^\top \\
&= \begin{bmatrix}-\frac{1}{4} & -\frac{3}{4} & -\frac{\sqrt{6}}{4}\\ -\frac{3}{4} & \frac{11}{20} & -\frac{3\sqrt{6}}{70}\\ -\frac{\sqrt{6}}{4} & -\frac{3\sqrt{6}}{70} & \frac{7}{10}\end{bmatrix} \\
HA &= \begin{bmatrix}-4 & * & *\\0 & * & *\\0 & * & *\end{bmatrix}
\end{aligned}
$$

Question 6

Verify that

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

Satisfies being an orthogonal matrix.
Note: Cannot use determinant to verify.

Sample Solutions

$$\begin{aligned}
A^\top A &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & \sin(\alpha) \\ 0 & -\sin(\alpha) & \cos(\alpha) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos^2(\alpha) + \sin^2(\alpha) & -\sin(\alpha)\cos(\alpha) + \sin(\alpha)\cos(\alpha) \\ 0 & -\sin(\alpha)\cos(\alpha) + \sin(\alpha)\cos(\alpha) & \sin^2(\alpha) + \cos^2(\alpha) \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

Thus A is an orthogonal matrix.

Question 7

What is the cosine of the angle between

$$\left(6, -6, -4, \sqrt{12}\right), \left(6, 4, 2, \sqrt{25}\right)?$$

My Solution

$$\begin{aligned}
\cos(\angle(\vec{a}, \vec{b})) &= \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|\|\vec{b}\|} \\
&= \frac{6 \cdot 6 + (-6) \cdot 4 + (-4) \cdot 2 + \sqrt{12} \cdot \sqrt{25}}{\sqrt{6^2 + (-6)^2 + (-4)^2 + \sqrt{12}^2}\sqrt{6^2 + 4^2 + 2^2 + \sqrt{25}^2}} \\
&= \frac{36 - 24 - 8 + 5\sqrt{12}}{\sqrt{100}\sqrt{81}} \\
&= \frac{4 + 5\sqrt{12}}{90} \\
\approx 0.24
\end{aligned}$$

Question 8

Another 3×3 affine system

- show the intermediate result when the first column is the one hot vector $\begin{bmatrix}1 & 0 & 0\end{bmatrix}$ for the first time
- show that the intermediate result when the matrix has row echelon form for the first time
- get the solution

$$\begin{aligned} 2x - 3y + 2z &= -4 \\ 7x + 4.5y - 1z &= 16 \\ 4x + 3y + z &= 2 \end{aligned}$$

My Solution

$$\begin{aligned}
& \left[\begin{array}{ccc|c} 2 & -3 & 2 & -4 \\ 7 & 4.5 & -1 & 16 \\ 4 & 3 & 1 & 2 \end{array}\right] \\
\frac{1}{2}\rho_1 \implies & \left[\begin{array}{ccc|c} 1 & -3/2 & 1 & -2 \\ 7 & 4.5 & -1 & 16 \\ 4 & 3 & 1 & 2 \end{array}\right] \\
(i)\rho_2 - 7\rho_1, \rho_3 - 4\rho_1 \implies & \left[\begin{array}{ccc|c} 1 & -3/2 & 1 & -2 \\ 0 & 15 & -8 & 30 \\ 0 & 9 & -3 & 10 \end{array}\right] \\
\frac{1}{15}\rho_2, -\frac{1}{3}\rho_3 \implies & \left[\begin{array}{ccc|c} 1 & -3/2 & 1 & -2 \\ 0 & 1 & -8/15 & 2 \\ 0 & -3 & 1 & -\frac{10}{3} \end{array}\right] \\
(ii)\rho_3 + 3\rho_2 \implies & \left[\begin{array}{ccc|c} 1 & -3/2 & 1 & -2 \\ 0 & 1 & -8/15 & 2 \\ 0 & 0 & 1 & -\frac{40}{9} \end{array}\right] \\
\rho_2 + \frac{8}{15}\rho_3 \implies & \left[\begin{array}{ccc|c} 1 & -3/2 & 1 & -2 \\ 0 & 1 & 0 & -\frac{10}{27} \\ 0 & 0 & 1 & -\frac{40}{9} \end{array}\right] \\
\rho_1 + \frac{3}{2}\rho_2 - \rho_3 \implies & \left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{17}{9} \\ 0 & 1 & 0 & -\frac{10}{27} \\ 0 & 0 & 1 & -\frac{40}{9} \end{array}\right] \\
\begin{bmatrix} x \\ y \\ z \end{bmatrix} = & \begin{bmatrix} -\frac{17}{9} \\ -\frac{10}{27} \\ -\frac{40}{9} \end{bmatrix}
\end{aligned}$$