INF 1004 Mathematics 2
Tutorial #9 Solutions

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Question 1

Draw these affine spaces

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} \cdot x + 1.5 = 0$$

$$\begin{bmatrix} -5 \\ -1 \end{bmatrix} \cdot x + 6 = 0$$

$$\begin{bmatrix} -5 \\ -1 \end{bmatrix} \cdot x - 6 = 0$$

My Solution

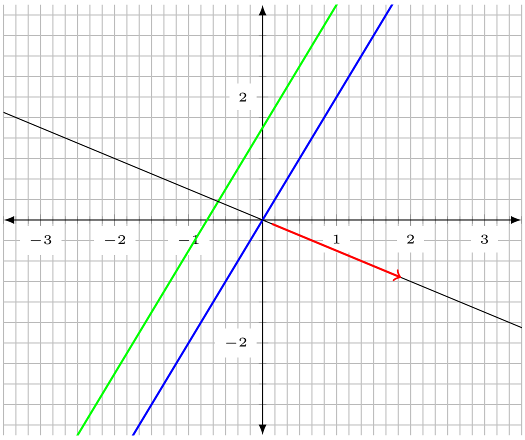
Part 1

Draw x where $w \cdot x = 0$

$\begin{pmatrix} 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} x_0 & x_1 \end{pmatrix} = 0 \quad \text{thus } x_1 = 2x_0$

Find $\frac{b}{w \cdot w} w = \frac{-1.5}{2^2 + (-1)^2} \begin{pmatrix} 2 & 1 \end{pmatrix} = -0.3 \begin{pmatrix} 2 & -1 \end{pmatrix} = \begin{pmatrix} -0.6 & 0.3 \end{pmatrix}$

Shift the original line by $\frac{b}{w \cdot w} w$



Verify $\begin{pmatrix} 2, -1 \end{pmatrix} \cdot \begin{pmatrix} x_0, x_1 \end{pmatrix} + 1.5$ is drawn

$2x_0 - x_1 + 1.5 = 0$ or $x_1 = 2x_0 + 1.5$

Looks like $y = 2x + 1.5$

Question 1 continued on next page. . .

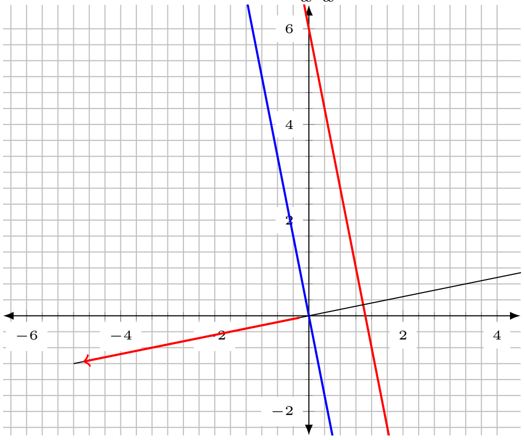
Part 2

Draw x where $w \cdot x = 0$

$\begin{pmatrix} -5 & -1 \end{pmatrix} \cdot \begin{pmatrix} x_0 & x_1 \end{pmatrix} = 0$ thus $x_1 = -5x_0$

Find $\frac{b}{w \cdot w} w = \frac{-6}{(-5)^2+(-1)^2} \begin{pmatrix} -5 & -1 \end{pmatrix} = -\frac{6}{26} \begin{pmatrix} -5 & -1 \end{pmatrix} = \begin{pmatrix} \frac{15}{13} & \frac{3}{13} \end{pmatrix}$

Shift the original line by $\frac{b}{w \cdot w} w$



Verify $\left(-5, -1\right) \cdot \left(x_0, x_1\right) + 6$ is drawn

$-5x_0 - x_1 + 6 = 0$ or $x_1 = -5x_0 + 6$

Looks like $y = -5x + 6$

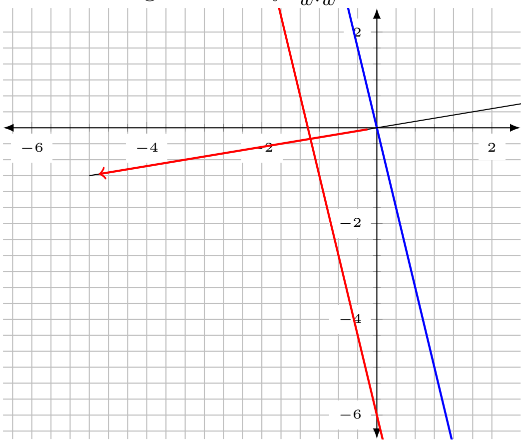
Part 3

Draw x where $w \cdot x = 0$

$\begin{pmatrix} -5 & -1 \end{pmatrix} \cdot \begin{pmatrix} x_0 & x_1 \end{pmatrix} = 0$ thus $x_1 = -5x_0$

Find $\frac{b}{w \cdot w} w = \frac{6}{(-5)^2+(-1)^2} \begin{pmatrix} -5 & -1 \end{pmatrix} = \frac{6}{26} \begin{pmatrix} -5 & -1 \end{pmatrix} = \begin{pmatrix} -\frac{15}{13} & -\frac{3}{13} \end{pmatrix}$

Shift the original line by $\frac{b}{w \cdot w} w$



Verify $\left(-5, -1\right) \cdot \left(x_0, x_1\right) - 6$ is drawn

$-5x_0 - x_1 - 6 = 0$ or $x_1 = -5x_0 - 6$

Looks like $y = -5x - 6$

Question 2

Find two non-parallel vectors x solving

$$w \cdot x = 3$$

$$w = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

My Solution

$$\begin{aligned}
w \cdot x = 3 &\implies \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = 3 \\
&\implies x_0 + 2x_1 + 4x_2 = 3 \\
&\implies x_0 = 3 + 2x_1 + 4x_2 (= 3 + 2s - 4t) \\
&\implies x_1 = x_1 (= s) \\
&\implies x_2 = x_2 (= t)
\end{aligned}$$

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - t \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$$

$$s = 1, t = 0 : \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$$

$$s = 0, t = 1 : \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$$

They are clearly linear independent. There is no c such that

$$\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} = c \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$$

$$\boxed{\text{Answer: vectors are: } \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}}$$

Question 3

Show that the line given by

$$f(t) = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$$

does not intersect the plane given by

$$2x + z = 9$$

Note:

$$f(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

Sample Solutions

Steps:

- Plug in $f(t)$ into your equation (system).
- Write the resulting equations as an affine system in $t : at = b$.
- Solve it for t.

$2x + z = 9$ take the x and z componenets

$$\begin{aligned} f(t) &= \begin{bmatrix} 2 + t(-1) \\ 3 + t(4) \\ 1 + t(2) \end{bmatrix} \\ x(t) &= 2 + t(-1) \\ z(t) &= 1 + 2t \\ \implies 2x + z &= 2(2 - t) + (1 + 2t) = 4 - 2t + 1 + 2t = 5 \end{aligned}$$

Cannot satisfy $5 = 9$ for any t. It is an affine equation.
Note: for the equation $2x + z = 5$ any t solves.
Therefore: The line lies inside the plane given by $2x + z = 5$.

$\boxed{\text{Answer: Showed that the line does not intersect plane.}}$

Question 4

Show that the line given by

$$f(t) = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + t \begin{bmatrix} 2 \\ 3 \\ -5 \end{bmatrix}$$

does not intersect the plane given by

$$3x - 2y + 2z = 18$$

Note:

$$f(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

Sample Solutions

$$\begin{aligned}
3x - 2y + 2z = 18 \text{ take the x, y and z components} \\
f(t) &= \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} \\
x(t) &= 1 + 2t \\
y(t) &= -3 + 3t \\
z(t) &= 2 - 5t \\
\implies 3x - 2y + 2z &= 3(1 + 2t) - 2(-3 + 3t) + 2(2 - 5t) = 3 + 6t + 6 - 6t + 4 - 10t = 13 - 10t = 18
\end{aligned}$$

Solve $13 - 10t = 18$ for t. It is an affine equation.
$-10t = 5$
$t = -0.5$

$$\implies t = -0.5 \implies \text{ plug it in } f(-0.5) = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} - 0.5 \begin{bmatrix} 2 \\ 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ -4.5 \\ 4.5 \end{bmatrix}$$

Verify: $3x - 2y + 2z = 3 \cdot 0 - 2 \cdot (-4.5) + 2 \cdot 4.5 = 18$

$$\boxed{\text{Answer: Showed that the line intersects plane at } t = -0.5.}$$

Question 5

Check whether the plane given by

$$f(s,t) = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

has an intersection with the line given by

$$x + 2y - z = 3$$
$$2x - y + z = 6$$

Note:

$$f(s,t) = \begin{bmatrix} x(s,t) \\ y(s,t) \\ z(s,t) \end{bmatrix}$$

Sample Solutions

Steps:

- Plug in $f(s,t)$ into your equation (system).
- Write the resulting equations as an affine system in vector $\begin{bmatrix} s \\ t \end{bmatrix}$: $A \begin{bmatrix} s \\ t \end{bmatrix} = b$.
- Solve it for t.

$$f(s,t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

$$x(s,t) = 0 + t(-2) + s(1)$$
$$y(s,t) = 1 + t(0) + s(2)$$
$$z(s,t) = -3 + t(1) + s(-1)$$

$$x + 2y - z = -2t + s + 2(1 + 2s) - (-3 + t - s) = 5 - 2t - t + s + 4s + s = 5 + 6s - 3t$$
$$2x - y + z = 2(-2t + s) - (1 + 2s) + -3 + t - s = -4 - 4t + t + 2s - 2s - s = -4 - s - 3t$$
$$5 + 6s - 3t = 3$$
$$-4 - s - 3t = 6$$

$$\left[\begin{array}{cc|c} 6 & -3 & -2 \\ -1 & -3 & 10 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 3 & -10 \\ 6 & -3 & -2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 3 & -10 \\ 0 & -21 & 58 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 3 & -10 \\ 0 & 1 & -58/21 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & -\frac{12}{7} \\ 0 & 1 & -\frac{58}{21} \end{array}\right]$$

$\boxed{\text{Answer: Intersects where } s = -\frac{12}{7} \text{ and } t = -\frac{58}{21}.}$

Question 6

Convert the plane equation into the form $Ax = b$ for

$$\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ -3 \end{bmatrix} + s \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

Steps:

- What is the dimensionality of the whole vector space in which these equations are defined?
- What is the dimensionality of the affine space spanned by the plane equation?
- How does the matrix B look like for which we seek solutions x such that $Bx = 0$?
- Conclude based on the dimensionality of the whole vector space and the dimensionality of the plane, what is the dimensionality of solutions x which we are searching for ?
- Find a basis for these solutions. Turn it into a matrix A
- Get the correct bias vector b based the A which you found

Part 1

The whole space is 3 dim

Part 2

$v_0 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $v_1 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$

are linearly independent, therefore they span a 2-dim space (k=2)

Question 6 continued on next page. . .

Part 3

Need $v_0 \cdot w = 0, v_1 \cdot w = 0$
$\begin{pmatrix}1 & 2 & 3\end{pmatrix} \cdot w = 0$
$\begin{pmatrix}2 & -1 & 2\end{pmatrix} \cdot w = 0$

$$x + 2y + 3z = 0$$
$$2x + -y + 2z = 0$$
$$\begin{bmatrix}1 & 2 & 3\\2 & -1 & 2\end{bmatrix} \cdot \begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$
$$\left[\begin{array}{ccc|c}1 & 2 & 3 & 0\\2 & -1 & 2 & 0\end{array}\right] \rightarrow \left[\begin{array}{ccc|c}1 & 0 & \frac{7}{5} & 0\\0 & 1 & \frac{4}{5} & 0\end{array}\right]$$

$$x + \frac{7}{5}z = 0$$
$$y + \frac{4}{5}z = 0$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = z\begin{bmatrix}-\frac{7}{5}\\-\frac{4}{5}\\1\end{bmatrix}$$

Part 4

We need to find one orthogonal vector $w = \begin{bmatrix}x\\y\\z\end{bmatrix}$ which is orthorgonal because then we have 3 vectors. 3 vectors span a 3-dim space. $(e - k = 1)$

Part 5

Let $z = 1$:

$$w = \begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}-\frac{7}{5}\\-\frac{4}{5}\\1\end{bmatrix}$$

$Ax = b$
To find A and B

$$A = \begin{bmatrix}-\frac{7}{5} & -\frac{4}{5} & 1\end{bmatrix}$$

Part 6

$$b = A\begin{bmatrix}5\\-2\\-3\end{bmatrix} = \begin{bmatrix}-\frac{7}{5} & -\frac{4}{5} & 1\end{bmatrix}\begin{bmatrix}5\\-2\\-3\end{bmatrix} = -8.4$$

Part 7

$$b = A\begin{bmatrix}5\\-2\\-3\end{bmatrix} = \begin{bmatrix}-\frac{7}{5} & -\frac{4}{5} & 1\end{bmatrix}\begin{bmatrix}5\\-2\\-3\end{bmatrix} = -8.4$$

Question 7

Convert the plane equation into the form $Ax = b$ for

$$\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} -3 \\ 1 \\ 6 \end{bmatrix} + t\begin{bmatrix} 2 \\ -4 \\ -4 \end{bmatrix}$$

Sample Solutions

We have a 2 dimensinal affine space, spanned by:

$$\begin{bmatrix} -3 \\ 1 \\ 6 \end{bmatrix} \quad \begin{bmatrix} 2 \\ -4 \\ -4 \end{bmatrix}$$

There can be in 3 dimensionsonly one orthogonal vector w which is orthogonal to both these vectors
$\begin{bmatrix} -3 & 1 & 6 \\ 2 & -4 & -4 \end{bmatrix} \cdot w = 0$
$\begin{bmatrix} -3 & 1 & 6 \\ 2 & -4 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \end{bmatrix}$

$y = 0$
$x - 2z = 0$

$$w = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = z\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

$w = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and thus $A = \begin{bmatrix} 2 & 0 & -1 \end{bmatrix}$

$b = \begin{bmatrix} 2 & 0 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = -1$

Question 8

Plot 2d plandes in a 3d space using e.g. matplotlib

My Solution