

INF 1004 Mathematics 2
Tutorial #7 Solution

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Question 1

Compute the euclidean vector norm for vectors

$$[1, 0, 2], [3, 4], [-7, 2, -4, \sqrt{12}]$$

My Solution

Part 1

$$\|[1, 0, 2]\|_2 = \sqrt{1^2 + 0^2 + 2^2} = \boxed{\sqrt{5}}$$

Part 2

$$\|[3, 4]\|_2 = \sqrt{3^2 + 4^2} = \sqrt{25} = \boxed{5}$$

Part 3

$$\begin{aligned}\|[-7, 2, -4, \sqrt{12}]\|_2 &= \sqrt{(-7)^2 + 2^2 + (-4)^2 + \sqrt{12}^2} \\ &= \sqrt{49 + 4 + 16 + 12} \\ &= \sqrt{81} \\ &= \boxed{9}\end{aligned}$$

Question 2

Compute the corresponding unit length vector for these:

$$[3, 4], [-1, -2, 3], [-7, 2, -4, \sqrt{12}]$$

My Solution

$$\begin{aligned}\|v\|_2 &= 1 \\ v \neq 0 &\implies \frac{v}{\|v\|_2}\end{aligned}$$

Part 1

$$\begin{aligned}\frac{[3, 4]}{\|[3, 4]\|_2} &= \frac{[3, 4]}{5} \\ &= \frac{1}{5} [3, 4] \\ &= \boxed{\left[\frac{3}{5}, \frac{4}{5}\right]}\end{aligned}$$

Part 2

$$\begin{aligned}\frac{[-1, -2, 3]}{\|[-1, -2, 3]\|_2} &= \frac{[-1, -2, 3]}{\sqrt{1^2 + 4^2 + 9^2}} \\ &= \frac{1}{\sqrt{14}} [-1, -2, 3] \\ &= \boxed{\left[-\frac{1}{\sqrt{14}}, -\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right]}\end{aligned}$$

Part 3

$$\begin{aligned}\frac{[-7, 2, -4, \sqrt{12}]}{\|[-7, 2, -4, \sqrt{12}]\|_2} &= \frac{[-7, 2, -4, \sqrt{12}]}{\sqrt{(-7)^2 + 2^2 + (-4)^2 + (\sqrt{12})^2}} \\ &= \frac{1}{\sqrt{81}} [-7, 2, -4, \sqrt{12}] \\ &= \frac{1}{9} [-7, 2, -4, \sqrt{12}] \\ &= \boxed{\left[-\frac{7}{9}, \frac{2}{9}, -\frac{4}{9}, \frac{\sqrt{12}}{9}\right]}\end{aligned}$$

Question 3

Compute the inner product between these vectors and their angle in degrees:

$$[3, -2, 2], [1, 2, 2]$$

Compute the inner product between these vectors and their angle in degrees:

$$[1, 0, 1], [2, 1, -2], \left[\frac{1}{2\sqrt{2}}, -\frac{\sqrt{3}}{2}, \frac{1}{2\sqrt{2}} \right]$$

Sample Solutions

$$\frac{u}{\|u\|_2} \cdot \frac{v}{\|v\|_2} = \cos(\angle(u, v))$$

Part 1

$$\begin{aligned} \langle [3, -2, 2], [1, 2, 2] \rangle &= 3 \cdot 1 + (-2) \cdot 2 + 2 \cdot 2 \\ &= 3 + (-4) + 4 \\ &= \boxed{3} \end{aligned}$$

Angle:

Let u be $[3, -2, 2]$ and v be $[1, 2, 2]$

$$\begin{aligned} \cos(\angle(u, v)) &= \frac{\langle u, v \rangle}{\|u\|_2 \cdot \|v\|_2} \\ &= \frac{[3, -2, 2] \cdot [1, 2, 2]}{\|[3, -2, 2]\|_2 \cdot \|[1, 2, 2]\|_2} \\ &= \frac{3}{\sqrt{17} \cdot \sqrt{9}} \\ &= \frac{3}{\sqrt{153}} \\ &= \frac{1}{\sqrt{17}} \\ \angle(u, v) &= \cos^{-1}\left(\frac{3}{\sqrt{153}}\right) \\ &= \boxed{75.96^\circ} \text{ OR} \\ &= 360^\circ - 75.96^\circ \\ &= \boxed{284.04^\circ} \end{aligned}$$

Part 2

$$\begin{aligned}
 \langle [1, 0, 1], [2, 1, -2] \rangle &= 1 \cdot 2 + 0 \cdot 1 + 1 \cdot (-2) \\
 &= 2 + 0 + (-2) \\
 &= \boxed{0}
 \end{aligned}$$

Angle:

Let u be $[1, 0, 1]$ and v be $[2, 1, -2]$ and w be $\left[\frac{1}{2\sqrt{2}}, -\frac{\sqrt{3}}{2}, \frac{1}{2\sqrt{2}}\right]$

$$\begin{aligned}
 \cos(\angle(u, v)) &= \frac{\langle u, v \rangle}{\|u\|_2 \cdot \|v\|_2} \\
 &= \frac{[1, 0, 1] \cdot [2, 1, -2]}{\|[1, 0, 1]\|_2 \cdot \|[2, 1, -2]\|_2} \\
 &= \frac{0}{\sqrt{2} \cdot \sqrt{6}} \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 \angle(u, v) &= \cos^{-1}(0) \\
 &= \boxed{90^\circ} \text{ OR} \\
 &= 360^\circ - 90^\circ \\
 &= \boxed{270^\circ}
 \end{aligned}$$

Part 3

$$\left\langle [1, 0, 1] \cdot \left[\frac{1}{2\sqrt{2}}, -\frac{\sqrt{3}}{2}, \frac{1}{2\sqrt{2}} \right] \right\rangle = \frac{1}{\sqrt{2}}$$

$$\left\| \left[\frac{1}{2\sqrt{2}}, -\frac{\sqrt{3}}{2}, \frac{1}{2\sqrt{2}} \right] \right\| = \sqrt{\frac{1}{4 \cdot 2} + \frac{3}{4} + \frac{1}{4 \cdot 2}} = 1$$

$$\begin{aligned}
 \cos \angle(u, w) &= \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{1} \\
 &= \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 \angle(u, w) &= \cos^{-1}\left(\frac{1}{2}\right) \\
 &= 60^\circ \text{ OR}
 \end{aligned}$$

$$\angle(u, w) = 360^\circ - 60^\circ = 300^\circ$$

Part 4

$$\begin{aligned} [2, 1, -2] \cdot \left[\frac{1}{2\sqrt{2}}, -\frac{\sqrt{3}}{2}, \frac{1}{2\sqrt{2}} \right] &= \frac{1}{2\sqrt{2}} \cdot 2 + \frac{\sqrt{3}}{2} \cdot 1 + \frac{1}{2\sqrt{2}} \cdot (-2) \\ &= -\frac{\sqrt{3}}{2} \end{aligned}$$

$$\begin{aligned} \cos \angle(v, w) &= -\frac{\sqrt{3}}{2} \cdot \frac{1}{3} \cdot \frac{1}{1} \\ &= -\frac{1}{2\sqrt{3}} \end{aligned}$$

$$\begin{aligned} \angle(v, w) &= \cos^{-1}\left(-\frac{1}{2\sqrt{3}}\right) \\ &= 106.74^\circ \text{ OR} \end{aligned}$$

$$\angle(v, w) = 360^\circ - 106.74^\circ = 253.26^\circ$$

Question 4

- What is the projection of $[5, 2]$ onto the subspace spanned by vector $[1, 1]$?
- What is the projection of $[0, 2, 1]$ onto the subspace spanned by vector $[1, -1, -1]$?
- Project $[5, 2]$ onto the subspace spanned by vectors $[2, 3], [1, 1]$
- What is the projection of $[1, -1, 1]$ ~~onto the subspace spanned by vector $[1, 1, 1]$~~ onto the subspace spanned by vectors $[0, 0, -1], [2, 0, 1]$? Hint: this one is more tricky. Reason: $[0, -1, -1] \cdot [2, 0, 1] \neq 0$

My Solution

$$x_{\parallel v} = \frac{x \cdot v}{v \cdot v} v = \left(x \cdot \frac{v}{\|v\|_2} \right) \frac{v}{\|v\|_2}$$

Part 1

$$\begin{aligned} \frac{[5, 2] \cdot [1, 1]}{2} [1, 1] &= \frac{7}{2} [1, 1] \\ &= [3.5, 3.5] \end{aligned}$$

Part 2

$$\begin{aligned} \frac{[0, 2, 1] \cdot [1, -1, -1]}{3} [1, -1, -1] &= \frac{-3}{3} [1, -1, -1] \\ &= [-1, 1, 1] \end{aligned}$$

Part 3

Use Gram Schmidts to get 2 orthogonal Vectors $\tilde{v}^{\{0\}}$ and $\tilde{v}^{\{1\}}$

Project $[5, 2]$ onto the subspace spanned by $\tilde{v}^{\{0\}}$ and $\tilde{v}^{\{1\}}$ and sum up projections

$$\begin{aligned} \tilde{v}^{\{0\}} &= [1, 1] \\ \tilde{v}^{\{1\}} &= [2, 3] \\ \tilde{v}^{\{1\}} &= [2, 3] - \frac{[2, 3] \cdot [1, 1]}{[1, 1] \cdot [1, 1]} [1, 1] \\ &= [2, 3] - \frac{5}{2} [1, 1] \\ &= \left[-\frac{1}{2}, \frac{1}{2}\right] \end{aligned}$$

Projection:

$$\begin{aligned} \frac{[5, 2] \cdot [1, 1]}{2} [1, 1] + \frac{[5, 2] \cdot \left[-\frac{1}{2}, \frac{1}{2}\right]}{\left[-\frac{1}{2}, -\frac{1}{2}\right]} \left[-\frac{1}{2}, \frac{1}{2}\right] &= \frac{7}{2} [1, 1] + \frac{-\frac{3}{2}}{\frac{1}{2}} \left[-\frac{1}{2}, \frac{1}{2}\right] \\ &= [3.5, 3.5] + \left[-\frac{3}{2}, \frac{3}{2}\right] \\ &= \left[3.5 - \frac{3}{2}, 3.5 + \frac{3}{2}\right] \\ &= [5, 2] \end{aligned}$$

It must be $[5, 2]$.

Reason: $[2, 3] \cdot [1, 1]$ are 2 independent vectors \mathbb{R}^2 span the whole vector space.

2 independent vectors a,b, therefore dimension of subspace spanned by a and b is \mathbb{R}^2
[5, 2] projected onto \mathbb{R}^2 is [5, 2]

Part 4

$$\text{let } \begin{cases} v^{\{0\}} = [0, 0, -1] \\ v^{\{1\}} = [2, 0, 1] \end{cases}$$

$$\begin{aligned} v^{\{1\}} &= [2, 0, 1] - \frac{[2, 0, 1] \cdot [0, 0, -1]}{[0, 0, -1] \cdot [0, 0, -1]} [0, 0, -1] \\ &= [2, 0, 1] - \frac{1}{1} [0, 0, -1] \\ &= [2, 0, 1] - [0, 0, -1] \\ &= [2, 0, 0] \end{aligned}$$

Now project $[1, -1, 1]$ onto the subspace spanned by $v^{\{0\}}$ and $v^{\{1\}}$

$$\begin{aligned} x_{\parallel v} &= \frac{[1, -1, 1] \cdot [0, 0, -1]}{[0, 0, -1] \cdot [0, 0, -1]} [0, 0, -1] + \frac{[1, -1, 1] \cdot [2, 0, 0]}{[2, 0, 0] \cdot [2, 0, 0]} [2, 0, 0] \\ &= \frac{-1}{1} [0, 0, -1] + \frac{2}{4} [2, 0, 0] \\ &= [1, 0, 1] \end{aligned}$$

Question 5

Compute these matrix multiplications:

$$\begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ -4 & -2 \end{bmatrix}$$

$$\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} [2 \quad 4 \quad -2]$$

Question: Do you need more of them to practice? If so, you can do at home:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 0.5 & 2.5 \\ -3.5 & 1.5 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

My Solution

Part 1

$$\begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ -4 & -2 \end{bmatrix} = \begin{bmatrix} 2 \cdot (-1) + 1 \cdot (-4) & 2 \cdot 0 + 1 \cdot (-2) \\ 3 \cdot (-1) + (-2) \cdot (-4) & 3 \cdot 0 + (-2) \cdot (-2) \end{bmatrix} = \begin{bmatrix} -6 & -2 \\ 5 & 4 \end{bmatrix}$$

$$\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} [2 \quad 4 \quad -2] = \begin{bmatrix} -3 \cdot 2 & -3 \cdot 4 & -3 \cdot (-2) \\ 2 \cdot 2 & 2 \cdot 4 & 2 \cdot (-2) \\ 1 \cdot 2 & 1 \cdot 4 & 1 \cdot (-2) \end{bmatrix} = \begin{bmatrix} -6 & -12 & 6 \\ 4 & 8 & -4 \\ 2 & 4 & -2 \end{bmatrix}$$

Part 2

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 \cdot 3 + 2 \cdot 0 & 1 \cdot 0 + 2 \cdot 1 & 1 \cdot 1 + 2 \cdot 2 \\ 2 \cdot 3 + 4 \cdot 0 & 2 \cdot 0 + 4 \cdot 1 & 2 \cdot 1 + 4 \cdot 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 5 \\ 6 & 4 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 0.5 & 2.5 \\ -3.5 & 1.5 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 0.5 \cdot 6 + 2.5 \cdot 4 \\ 0.5 \cdot 4 + 2.5 \cdot 6 \end{bmatrix} = \begin{bmatrix} 13 \\ -15 \end{bmatrix}$$

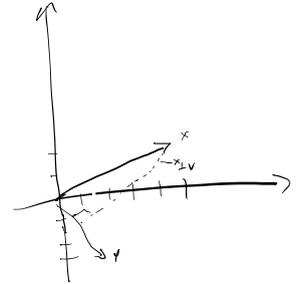
Question 6

- Project $[5, 2]$ onto the orthogonal space of vector $[2, -3]$
- Project $[1, -1, 3]$ onto the orthogonal space of vector $[-3, 1, 1]$
- Project $[1, -1, 3, 1]$ onto the orthogonal space of vector $[-2, 2, 0, 0]$, $[0, 0, \sqrt{2}, \sqrt{2}]$

My Solution

Part 1

$$\begin{aligned}
 x_{\perp v} &= [5, 2] - \frac{[5, 2] \cdot [2, -3]}{[2, -3] \cdot [2, -3]} [2, -3] \\
 &= [5, 2] - \frac{5 \cdot 2 + 2 \cdot (-3)}{2 \cdot 2 + (-3) \cdot (-3)} [2, -3] \\
 &= [5, 2] - \frac{4}{13} [2, -3] \\
 &= \left[5 - \frac{8}{13}, 2 + \frac{12}{13}\right] \\
 &= \left[\frac{57}{13}, \frac{38}{13}\right]
 \end{aligned}$$



Verify:

$$\left[5 - \frac{8}{13}, 2 + \frac{12}{13}\right] \cdot [2, -3] = 0$$

Part 2

$$\begin{aligned}
 x_{\perp v} &= [1, -1, 3] - \frac{[1, -1, 3] \cdot [-3, 1, 1]}{[-3, 1, 1] \cdot [-3, 1, 1]} [-3, 1, 1] \\
 &= [1, -1, 3] - \frac{1 \cdot (-3) + (-1) \cdot 1 + 3 \cdot 1}{(-3) \cdot (-3) + 1 \cdot 1 + 1 \cdot 1} [-3, 1, 1] \\
 &= [1, -1, 3] - \frac{-1}{11} [-3, 1, 1] \\
 &= \left[1 - \frac{3}{11}, -1 + \frac{1}{11}, 3 + \frac{1}{11}\right] \\
 &= \left[\frac{8}{11}, -\frac{10}{11}, \frac{34}{11}\right]
 \end{aligned}$$

Verify:

$$\left[1 - \frac{3}{11}, -1 + \frac{1}{11}, 3 + \frac{1}{11}\right] \cdot [-3, 1, 1] = 0$$

Part 3

$$[-2, 2, 0, 0] \cdot [0, 0, \sqrt{2}, \sqrt{2}] = 0, \text{ thus:}$$

$$\begin{aligned}x_{\perp v} &= [1, -1, 3, 1] - \frac{[1, -1, 3, 1] \cdot [-2, 2, 0, 0]}{[-2, 2, 0, 0] \cdot [-2, 2, 0, 0]} [-2, 2, 0, 0] - \frac{[1, -1, 3, 1] \cdot [0, 0, \sqrt{2}, \sqrt{2}]}{[0, 0, \sqrt{2}, \sqrt{2}] \cdot [0, 0, \sqrt{2}, \sqrt{2}]} [0, 0, \sqrt{2}, \sqrt{2}] \\&= [1, -1, 3, 1] - \frac{-4}{8} [-2, 2, 0, 0] - \frac{0}{2} [0, 0, \sqrt{2}, \sqrt{2}] \\&= [1, -1, 3, 1] - [-1, 1, 0, 0] - [0, 0, 2, 2] \\&= [0, 0, 1, -1]\end{aligned}$$

Verify:

Verify that $[0, 0, 1, -1] \cdot [-2, 2, 0, 0] = 0$ and $[0, 0, 1, -1] \cdot [0, 0, \sqrt{2}, \sqrt{2}] = 0$.

Question 7

Run Gram-Schmid-orthogonalization on the vectors

$$[12, 12, 6], [2, -2, 4], [-2, -2, 1]$$

My Solution

Part 1

$$\tilde{v}^{\{0\}} = [12, 12, 6]$$

$$\begin{aligned} \tilde{v}^{\{1\}} &= [2, -2, 4] - \frac{[2, -2, 4] \cdot [12, 12, 6]}{[12, 12, 6] \cdot [12, 12, 6]} [12, 12, 6] \\ &= [2, -2, 4] - \frac{2 \cdot 12 + (-2) \cdot 12 + 4 \cdot 6}{12 \cdot 12 + 12 \cdot 12 + 6 \cdot 6} [12, 12, 6] \\ &= [2, -2, 4] - \frac{24}{288 + 36} [12, 12, 6] \\ &= [2, -2, 4] - \left[\frac{12 \cdot 24}{9 \cdot 36}, \frac{12 \cdot 24}{9 \cdot 36}, \frac{6 \cdot 24}{9 \cdot 36} \right] \\ &= [2, -2, 4] - \left[\frac{288}{324}, \frac{288}{324}, \frac{144}{324} \right] \\ &= \left[2 - \frac{8}{9}, -2 - \frac{8}{9}, 4 - \frac{4}{9} \right] \end{aligned}$$

$$\begin{aligned} \tilde{v}^{\{2\}} &= [-2, -2, 1] \\ &\quad - \frac{[-2, -2, 1] \cdot [12, 12, 6]}{[12, 12, 6] \cdot [12, 12, 6]} [12, 12, 6] \\ &\quad - \frac{[-2, -2, 1] \cdot \left[2 - \frac{8}{9}, -2 - \frac{8}{9}, 4 - \frac{4}{9} \right]}{\left[2 - \frac{8}{9}, -2 - \frac{8}{9}, 4 - \frac{4}{9} \right] \cdot \left[2 - \frac{8}{9}, -2 - \frac{8}{9}, 4 - \frac{4}{9} \right]} \left[2 - \frac{8}{9}, -2 - \frac{8}{9}, 4 - \frac{4}{9} \right] \\ &= [-2, -2, 1] - \frac{(-2) \cdot 12 + (-2) \cdot 12 + 1 \cdot 6}{12 \cdot 12 + 12 \cdot 12 + 6 \cdot 6} [12, 12, 6] \\ &\quad - \frac{(-2) \cdot 2 + (-2) \cdot (-2) + 1 \cdot 4}{2 \cdot 2 + (-2) \cdot (-2) + 4 \cdot 4} \left[2 - \frac{8}{9}, -2 - \frac{8}{9}, 4 - \frac{4}{9} \right] \\ &= [-2, -2, 1] - \left[\frac{14}{9}, \frac{14}{9}, \frac{7}{9} \right] - \frac{\frac{36+28}{9}}{24 + \frac{128}{28} - \frac{32 \cdot 9}{81}} \left[2 - \frac{8}{9}, -2 - \frac{8}{9}, 4 - \frac{4}{9} \right] \end{aligned}$$

Question 8

Understanding Distances coming from ℓ_p -norms

Coding: plot in python or similar the set of points $x \in \mathbb{R}^2$ usuch that $\|x\|_p = 1$ for

- $p = 0.2$
- $p = 0.5$
- $p = 1$
- $p = 1.5$
- $p = 2$
- $p = 4$
- $p = 8$
- $p = 16$

Hint: in 2 dimensions for $p = 2$ the solution is given by

$$x(t) = (\cos(t), \sin(t))$$

due to $\cos^2(t) + \sin^2(t) = 1$.

you can use the same idea with different powers. You can start by considering $(\cos^r(t), \sin^r(t))$. One thing to note: $\cos(t)^r + \sin(t)^r$ is not always defined for negative values and certain r .

For $p \neq 2$, you can consider this, which deals with the signs:

$$x(t) = (\text{sign}(\cos(t))|\cos(t)|^r, \text{sign}(\sin(t))|\sin(t)|^r)$$

for the right choice of r . Find out which r is suitable for a general $p > 0$ such that $\|x\|_p = 1$. Then plot it in python.

My Solution

