Linear Algebra L1 - Vectors

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Learning Goals

- vectors of real numbers
- norms of vectors and their properties
- inner products, their interpretation and properties
- representing a vector as a linear combination
- vector spaces
- independent sets of vectors
- orthogonal sets of vectors
- projecting onto a vector, removing the direction of a vector
- creating an orthogonal set of vectors

Task 1

Compute the euclidean vector norm for vectors

$$[1,0,2],\ [3,4],\ [-7,2,-4,\sqrt{12}]$$

$$\begin{aligned}\sqrt{1^2+2^2} &= \sqrt{5}\\ \sqrt{3^2+3^2} &= \sqrt{25} = 5\\ \sqrt{(-7)^2+2^2+(-4)^2+(\sqrt{12})^2} &= \sqrt{49+4+16+12} = \sqrt{81} = 9\end{aligned}$$

Task 2

Compute the corresponding unit length vector for these:

$$[3,4],\ [-1,-2,3],\ [-7,2,-4,\sqrt{12}]$$

$$\begin{aligned}&[3/5, 4/5]\\ &[-1/\sqrt{14}, -2/\sqrt{14}, 3/\sqrt{14}]\\ &[-7/9, 2/9, -4/9, \sqrt{12}/9]\end{aligned}$$

Task 3

Compute the inner product between these vectors and their angle in degrees:

$$[3,-2,2],[1,2,2]$$

$$u \cdot v = 3 - 4 + 4 = 3$$
$$\cos \angle(u,v) = \frac{3}{\sqrt{17}\sqrt{9}} = \frac{1}{\sqrt{17}}$$

Note the two solutions for the angle. The second solution is -1 times the first solution.

$$\Rightarrow \angle_0(u,v) \approx 1.3258 = 0.422\pi \sim 75.86deg$$
$$\angle_1(u,v) = -0.422\pi = 2\pi - 0.422\pi \sim -75.86deg = 360 - 75.86deg = 284.14deg$$

Compute the inner product between these vectors and their angle in degrees:

$$[1,0,1],[2,1,-2],[\frac{1}{2\sqrt{2}},-\frac{\sqrt{3}}{2},\frac{1}{2\sqrt{2}}]$$

$$[1,0,1]\cdot[2,1,-2]:\ 0, \angle_0(u,v) = 0$$
$$[1,0,1]\cdot[\frac{1}{2\sqrt{2}},-\frac{\sqrt{3}}{2},\frac{1}{2\sqrt{2}}] = \frac{1}{\sqrt{2}}$$
$$\|[\frac{1}{2\sqrt{2}},-\frac{\sqrt{3}}{2},\frac{1}{2\sqrt{2}}]\| = \sqrt{\frac{1}{4*2}+\frac{3}{4}+\frac{1}{4*2}} = 1$$
$$\cos \angle(u,v) = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\frac{1}{1} = 1/2$$
$$\angle_0(u,v) = 1/3\pi \sim 60deg$$
$$[2,1,-2]\cdot[\frac{1}{2\sqrt{2}},-\frac{\sqrt{3}}{2},\frac{1}{2\sqrt{2}}] = -\frac{\sqrt{3}}{2}$$
$$\cos \angle(u,v) = -\frac{\sqrt{3}}{2}\frac{1}{3}\frac{1}{1} = -\frac{1}{2\sqrt{3}}$$
$$\angle_0(u,v) \approx 0.407\pi, \angle_1(u,v) \approx 2\pi - 0.407\pi,$$

in python this is convenient:

```
import math
math.acos( uv / uu**0.5 / vv**0.5 )/math.pi
```

gives you the multiple of pi ... like the ≈ -0.407 above

Task 4

- What is the projection of $[5,2]$ onto the subspace spanned by vector $[1,1]$?
- What is the projection of $[0,2,1]$ onto the subspace spanned by vector $[1,-1,-1]$?
- Project $[5,2]$ onto the subspace spanned by vectors $[2,3],\ [1,1]$
- What is the projection of $[1,-1,1]$ onto the subspace spanned by vectors $[0,0,-1],\ [2,0,1]$? Hint: this one is more tricky. Reason: $[0,0,-1]\cdot[2,0,1] \neq 0$

$$x_{\|v} = \frac{[5,2]\cdot[1,1]}{[1,1]\cdot[1,1]}[1,1] = \frac{7}{2}[1,1] = [3.5,3.5]$$
$$x_{\|v} = \frac{[0,2,1]\cdot[1,-1,-1]}{[1,-1,-1]\cdot[1,-1,-1]}[1,-1,-1] = \frac{-3}{3}[1,-1,-1] = [-1,1,1]$$

it must be $[5,2]$

Reason: $[2,3] \cdot [1,1] \neq 0$ and 2 independent vectors in $\mathbb{R}^2$ span the whole vector space.

vspace3mm
The last one:

- Either get an orthogonal basis and project onto it
- or remove all components orthogonal to these two vectors

The way by getting an Orthogonal Basis:

$$[2,0,1] - \frac{[2,0,1] \cdot [0,0,-1]}{[0,0,-1] \cdot [0,0,-1]}[0,0,-1] = [2,0,1] - \frac{-1}{1}[0,0,-1] = [2,0,0]$$

$$[0,0,-1] \text{ and } [2,0,0]$$

are an orthogonal basis.

Now project $[1,-1,1]$ onto it:

$$x_\parallel = \frac{[1,-1,1] \cdot [0,0,-1]}{[0,0,-1] \cdot [0,0,-1]}[0,0,-1] + \frac{[1,-1,1] \cdot [2,0,0]}{[2,0,0] \cdot [2,0,0]}[2,0,0] = \frac{-1}{1}[0,0,-1] + \frac{2}{4}[2,0,0] = [1,0,1]$$

Task 5

compute these matrix multiplications:

$$\begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ -4 & -2 \end{bmatrix}$$
$$\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 2*-1+1*-4 & 2*0+1*-2 \\ 3*-1+-2*-4 & 3*0+-2*-2 \end{bmatrix} = \begin{bmatrix} -6 & -2 \\ 5 & 4 \end{bmatrix}$$

The next one results in a (3,3)-shape matrix

$$\begin{bmatrix} -3*2 & -3*4 & -3*-2 \\ 2*2 & 2*4 & 2*-2 \\ 2 & 4 & -2 \end{bmatrix}$$

Question: Do you need more of them to practice? If so, you can do at home:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$
$$\begin{bmatrix} 0.5 & 2.5 \\ -3.5 & 1.5 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 5 \\ 6 & 4 & 10 \end{bmatrix}$$
$$\begin{bmatrix} 13 \\ -15 \end{bmatrix}$$

Task 6

- Project $[5,2]$ onto the orthogonal space of vector $[2,-3]$
- Project $[1,-1,3]$ onto the orthogonal space of vector $[-3,1,1]$
- Project $[1,-1,3,1]$ onto the orthogonal space of vectors $[-2,2,0,0]$, $[0,0,\sqrt{2},\sqrt{2}]$

Project onto the orthogonal space of a vector $\sim$ remove the component belonging to that vector.

$$\begin{aligned}
[5,2]-\frac{[5,2]\cdot[2,-3]}{[2,-3]\cdot[2,-3]}[2,-3] &= [5,2]-\frac{4}{13}[2,-3]=[5-\frac{8}{13},2+\frac{12}{13}]\\
\text{verify: } [5-\frac{8}{13},2+\frac{12}{13}]\cdot[2,-3] &= 10-\frac{16}{13}-6-\frac{36}{13}=4-\frac{52}{13}=0\\
[1,-1,3]-\frac{[1,-1,3]\cdot[-3,1,1]}{[-3,1,1]\cdot[-3,1,1]}[-3,1,1] &= [1,-1,3]-\frac{-1}{11}[-3,1,1]=[1-\frac{3}{11},-1+\frac{1}{11},3+\frac{1}{11}]\\
\text{verify: } [1-\frac{3}{11},-1+\frac{1}{11},3+\frac{1}{11}]\cdot[-3,1,1] &= -3+\frac{9}{11}-1+\frac{1}{11}+3+\frac{1}{11}=-4+3+\frac{11}{11}=0\\
[-2,2,0,0]\cdot[0,0,\sqrt{2},\sqrt{2}] &= 0, thus:\\
[1,-1,3,1]-\frac{[1,-1,3,1]\cdot[-2,2,0,0]}{[-2,2,0,0]\cdot[-2,2,0,0]}[-2,2,0,0] &- \frac{[1,-1,3,1]\cdot[0,0,\sqrt{2},\sqrt{2}]}{[0,0,\sqrt{2},\sqrt{2}]\cdot[0,0,\sqrt{2},\sqrt{2}]}[0,0,\sqrt{2},\sqrt{2}]=\\
&= [1,-1,3,1]-\frac{-4}{8}[-2,2,0,0]-\frac{4\sqrt{2}}{4}[0,0,\sqrt{2},\sqrt{2}]\\
&= [1,-1,3,1]+[-1,1,0,0]-[0,0,2,2]=[0,0,1,-1]
\end{aligned}$$

Verification for the last one also works out.

Task 7

run Gram-Schmid-orthogonalization on the vectors

$$[12,12,6],[2,-2,4],[-2,-2,1]$$

To Note:

- Before you start this process, you can divide vectors by constants. For example

$$[12,12,6]\to[2,2,1]$$
$$[2,-2,4]\to[1,-1,2]$$

- if you have no other goal (e.g. getting orthogonal to subspaces as in later lectures), then you could reorder vectors as well

$$\begin{aligned}
v^{\{1\}} &= [2,-2,4]-\frac{[2,-2,4]\cdot[12,12,6]}{[12,12,6]\cdot[12,12,6]}[12,12,6]=[2,-2,4]-\frac{24}{288+36}[12,12,6]=[2,-2,4]-[\frac{12*24}{9*36},\frac{12*24}{9*36},\frac{6*24}{9*36}]\\
&= [2,-2,4]-[\frac{8}{9},\frac{8}{9},\frac{4}{9}]=[2-\frac{8}{9},-2-\frac{8}{9},4-\frac{4}{9}]\\
v^{\{2\}} &= [-2,-2,1]-\frac{[-2,-2,1]\cdot[12,12,6]}{[12,12,6]\cdot[12,12,6]}[12,12,6]-\frac{[-2,-2,1]\cdot[2-\frac{8}{9},-2-\frac{8}{9},4-\frac{4}{9}]}{[2-\frac{8}{9},-2-\frac{8}{9},4-\frac{4}{9}]\cdot[2-\frac{8}{9},-2-\frac{8}{9},4-\frac{4}{9}]}[2-\frac{8}{9},-2-\frac{8}{9},4-\frac{4}{9}]\\
&= [-2,-2,1]-\frac{-3*14}{9*36}[12,12,6]-\frac{0+\frac{32}{9}+4-\frac{4}{9}}{4+\frac{64}{81}-\frac{32}{9}+4+\frac{64}{81}+\frac{32}{9}+16+\frac{16}{81}-\frac{32}{9}}[2-\frac{8}{9},-2-\frac{8}{9},4-\frac{4}{9}]\\
&= [-2,-2,1]-[\frac{14}{9},\frac{14}{9},\frac{7}{9}]-\frac{\frac{36+28}{9}}{24+\frac{128}{81}-\frac{32*9}{81}}[2-\frac{8}{9},-2-\frac{8}{9},4-\frac{4}{9}]
\end{aligned}$$

Task 8: understanding distances coming from ℓ_p-norms

Coding: plot in python or similar the set of points $x \in \mathbb{R}^2$ such that $\|x\|_p = 1$ for

- $p = 0.2$
- $p = 0.5$
- $p = 1$
- $p = 1.5$
- $p = 2$
- $p = 4$
- $p = 8$
- $p = 16$

Hint: in 2 dimensions for $p = 2$ the solution is given by

$$x(t) = (\cos(t), \sin(t))$$

due to $\cos^2(t) + \sin^2(t)$ =1.
You can use the same idea with different powers. You can start by considering $(\cos^r(t), \sin^r(t))$. One thing to note: $\cos(t)^r, \sin(t)^r$ is not always defined for negative values and certain r.

For $p \neq 2$ you can consider this, which deals with the signs:

$$x(t) = (sign(\cos(t))|\cos(t)|^r, sign(\sin(t))|\sin(t)|^r)$$

for the right choice of r. Find out which r is suitable for a general $p > 0$ such that $\|x\|_p = 1$. Then plot it in python.
Solution:
use

$$x(t) = (sign(\cos(t))|\cos(t)|^r, sign(\sin(t))|\sin(t)|^r)$$

with $r = 2/p$ and matplotlib or the like.
Plot it and enjoy the shapes