Linear Algebra L4 - Matrices

Alexander Binder

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1 Learning Goals

- matrix properties

Task 1

Compute $A^\top A$ for

$$A = \begin{bmatrix} 2 & -3 \\ -6 & -9 \end{bmatrix}$$
$$A = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 0 & 2 \\ 4 & 3 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 40 & 48 \\ 48 & 90 \end{bmatrix}$$
$$A = \begin{bmatrix} 20 & 12 & 8 \\ 12 & 10 & 2 \\ 8 & 2 & 6 \end{bmatrix}$$

Task 2

Compute the inverse of

$$A_0 = \begin{bmatrix} -2 & -3 \\ -6 & -4 \end{bmatrix}$$
$$A_1 = \begin{bmatrix} 3 & 1 \\ -2 & 2 \end{bmatrix}$$

Use these inverses to solve

$$A_0 x = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$
$$A_1 x = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

$$A_0^{-1} = -\frac{1}{10} \begin{bmatrix} -4 & 3 \\ 6 & -2 \end{bmatrix}$$
$$x = A_0^{-1} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = -\frac{1}{10} \begin{bmatrix} -4 & 3 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.9 \\ -1.6 \end{bmatrix}$$

$$A_1^{-1} = \frac{1}{8}\begin{bmatrix} 2 & -1 \\ 2 & 3 \end{bmatrix}$$
$$x = A_1^{-1}\begin{bmatrix} 2 \\ -5 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 2 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ -5 \end{bmatrix} = \begin{bmatrix} 1.125 \\ -1.375 \end{bmatrix}$$

Note: it is not common to solve $Ax = b$ using matrix inversion.

Reasons:

- $Ax = b$ can be solvable when A is not invertible
- It is often slower / more costly see e.g. https://gregorygundersen.com/blog/2020/12/09/matrix-inversion/

Task 3

Compute the determinant of

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 2 \\ 4 & 3 & 1 \end{bmatrix}$$
$$A = \begin{bmatrix} 2 & -4 & 3 \\ 1 & 1 & 1 \\ 3 & 3 & 1 \end{bmatrix}$$
$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 2 \\ -1 & 3 & 2 \end{bmatrix}$$
$$A = \begin{bmatrix} 1 & 2 & 3 \\ -5 & -10 & -15 \\ 6 & 12 & 18 \end{bmatrix}$$

- Are they invertible?
- Which of them has full rank ? Which of them has lower rank and which one ?

$$\begin{aligned} 0,&\ \text{no} \\ -12,&\ \text{y} \\ 9,&\ \text{y} \\ 0,&\ \text{n} \end{aligned}$$

Ranks: 2 (not rank 3, and no vec generates all others as multiples),3 (must have full rank),3 (must have full rank) ,1 (one vec generates all others as multiples)

Task 4

For what value a the matrix is not invertible ?

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & a & 4 \\ -3 & 1 & 2 \end{bmatrix}$$

$$
\begin{aligned}
det(A) &= 1 * a * 2 + 2 * 4 * -3 + 1 * 2 * 1 - 1 * 4 * 1 - 2 * 2 * 2 - 1 * a * (-3) \\
&= 5a - 24 + 2 - 4 - 8 = 5a - 30 \\
det(A) &= 0 \Leftrightarrow a = 6
\end{aligned}
$$

Task 5

Compute and apply the Householder matrix which makes transforms the first column of to a multiple of the first one-hot vector $\begin{bmatrix}1\\0\\0\end{bmatrix}$ for

$$
A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 2 \\ 2 & 3 & 1 \end{bmatrix}
$$

and for (hint: here subtracting is nicer)

$$
A = \begin{bmatrix} 1 & -4 & 3 \\ 3 & 1 & 1 \\ \sqrt{6} & 3 & 1 \end{bmatrix}
$$

case 0:

$$
\begin{aligned}
u &= \begin{bmatrix}1\\2\\2\end{bmatrix} \pm \|(1,2,2)\| \begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}-2\\2\\2\end{bmatrix} \\
H &= I - \frac{2}{12} u u^\top = I - \frac{1}{6} \begin{bmatrix}-2\\2\\2\end{bmatrix} \begin{bmatrix}-2 & 2 & 2\end{bmatrix} = I - \frac{1}{6} \begin{bmatrix} 4 & -4 & -4 \\ -4 & 4 & 4 \\ -4 & 4 & 4 \end{bmatrix} \\
&= I + \frac{2}{3} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}
\end{aligned}
$$

check possible here: $H = H^\top, HH = I$

$$
HA = \begin{bmatrix} 3 & 2.67 & 1.67 \\ 0 & -0.67 & -0.67 \\ 0 & 2.33 & -1.67 \end{bmatrix}
$$

case 1:

$$
\begin{aligned}
u &= \begin{bmatrix}1\\3\\\sqrt{6}\end{bmatrix} \pm \|(1,3,\sqrt{6})\| \begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}-3\\3\\\sqrt{6}\end{bmatrix} \\
H &= I - \frac{2}{24} u u^\top = I - \frac{1}{12} \begin{bmatrix}-3\\3\\\sqrt{6}\end{bmatrix} \begin{bmatrix}-3 & 3 & \sqrt{6}\end{bmatrix} = I - \frac{1}{12} \begin{bmatrix} 9 & -9 & -3\sqrt{6} \\ -9 & 9 & 3\sqrt{6} \\ -3\sqrt{6} & 3\sqrt{6} & 6 \end{bmatrix} \\
&= I + \begin{bmatrix} -0.75 & 0.75 & 1/4\sqrt{6} \\ 0.75 & -0.75 & -1/4\sqrt{6} \\ 1/4\sqrt{6} & -1/4\sqrt{6} & -0.5 \end{bmatrix} = \begin{bmatrix} 0.25 & 0.75 & 1/4\sqrt{6} \\ 0.75 & 0.25 & -1/4\sqrt{6} \\ 1/4\sqrt{6} & -1/4\sqrt{6} & 0.5 \end{bmatrix}
\end{aligned}
$$

check possible here: $H = H^\top, HH = I$

$$HA = \begin{bmatrix} 4 & -1 + 3/4 + 3/4\sqrt{6} & 3/4 + 3/4 + 1/4\sqrt{6} \\ 0 & 3/4 * -4 + 1/4 - 3/4\sqrt{6} & 3/4 * 3 + 0.25 - 1/4\sqrt{6} \\ 0 & 1/4\sqrt{6} * (-4) - 1/4\sqrt{6} + 0.5 * 3 & 1/4\sqrt{6} * 3 - 1/4\sqrt{6} * 1 + 0.5 * 1 \end{bmatrix}$$

Task 6

Verify that

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

satisfies being an orthogonal matrix.

$$\begin{aligned} A^\top A &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & \sin(\alpha) \\ 0 & -\sin(\alpha) & \cos(\alpha) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos^2(\alpha) + \sin^2(\alpha) & -\sin(\alpha)\cos(\alpha) + \sin(\alpha)\cos(\alpha) \\ 0 & -\sin(\alpha)\cos(\alpha) + \sin(\alpha)\cos(\alpha) & \cos^2(\alpha) + \sin^2(\alpha) \end{bmatrix} \end{aligned}$$

Task 7 (extra)

Because some of you had troubles with it
What is the cosine of the angle between

$$(6, -6, -4, \sqrt{12}),\ (6, 4, 2, \sqrt{25})\ ?$$

$$\begin{aligned} v_0 \cdot v_1 &= 36 - 24 - 8 + \sqrt{300} \\ \|v_0\| &= \sqrt{36 + 36 + 16 + 12} = 10 \\ \|v_1\| &= \sqrt{36 + 16 + 4 + 25} = \sqrt{81} = 9 \\ \cos\angle &= (4 + \sqrt{300})/90 \approx 0.24 \end{aligned}$$

Task 8 (extra)

another 3x3 affine system

- show the intermediate result when the first column is the one hot vector $[1, 0, 0]$ for the first time
- show the intermediate result when the matrix has row echelon form for the first time
- get the solution

$$\begin{aligned} 2x - 3y + 2z &= -4 \\ 7x + 4.5y - 1z &= 16 \\ 4x + 3y + z &= 2 \end{aligned}$$

$$\begin{bmatrix} 2 & -3 & 2 & | & -4 \\ 7 & 4.5 & -1 & | & 16 \\ 4 & 3 & 1 & | & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3/2 & 1 & | & -2 \\ 7 & 4.5 & -1 & | & 16 \\ 4 & 3 & 1 & | & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3/2 & 1 & | & -2 \\ 7 & 4.5 & -1 & | & 16 \\ 0 & 9 & -3 & | & 2+8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3/2 & 1 & | & -2 \\ 0 & 15 & -8 & | & 30 \\ 0 & 9 & -3 & | & 10 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -3/2 & 1 & | & -2 \\ 0 & 1 & -8/15 & | & 2 \\ 0 & 9 & -3 & | & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3/2 & 1 & | & -2 \\ 0 & 1 & -8/15 & | & 2 \\ 0 & 0 & -3+72/15 & | & 10-18 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3/2 & 1 & | & -2 \\ 0 & 1 & -8/15 & | & 2 \\ 0 & 0 & 1 & | & -40/9 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -3/2 & 0 & | & -2+40/9 \\ 0 & 1 & 0 & | & 2-40/9*8/15 \\ 0 & 0 & 1 & | & -40/9 \end{bmatrix} = \begin{bmatrix} 1 & -3/2 & 0 & | & 22/9 \\ 0 & 1 & 0 & | & -10/27 \\ 0 & 0 & 1 & | & -40/9 \end{bmatrix}$$

$$\rightarrow= \begin{bmatrix} 1 & 0 & 0 & | & 22/9+3/2*(-10)/27 \\ 0 & 1 & 0 & | & -10/27 \\ 0 & 0 & 1 & | & -40/9 \end{bmatrix}$$

first time one-hot vector:

$$\begin{bmatrix} 1 & -3/2 & 1 & | & -2 \\ 0 & 15 & -8 & | & 30 \\ 0 & 9 & -3 & | & 10 \end{bmatrix}$$

non-reduced row echelon form for the first time:

$$\begin{bmatrix} 1 & -3/2 & 1 & | & -2 \\ 0 & 1 & -8/15 & | & 2 \\ 0 & 0 & 1 & | & -40/9 \end{bmatrix}$$

solution:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \approx \begin{bmatrix} 1.89 \\ -0.37 \\ -4.44 \end{bmatrix}$$