Linear Algebra L5 - Eigenvalues and eigenvectors

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1 Learning Goals

- Eigenvalues and Eigenvectors

Task 1

- Compute the eigenvalues and eigenvectors for the matrices below.
- For one of these matrices compute the the matrix P such that $P^{-1}DP = A$, and verify that PAP^{-1} is the diagonal matrix of the eigenvalues.

$$A = \begin{bmatrix} 4 & \sqrt{15} \\ \sqrt{15} & 2 \end{bmatrix}$$ for this one we do it together

$$A = \begin{bmatrix} 2 & -2 \\ -3 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -4 \\ 4 & -6 \end{bmatrix}$$

Task 2

- Compute an eigenvector for the eigenvalue $x = 3$ for the below $(3, 3)$-matrix. Note: nobody asks you to compute its characteristic polynomial or to get all of its eigenvalues (Prof did it).
- Validate that the found eigenvector v is indeed the correct one, that is , that $Av = 3v$ holds.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 2 & 4 & 4 \end{bmatrix}$$

Task 3

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

- Show that this matrix has only the eigenvalue 1, twice.
- Prove that there cannot exist any matrix P such that $P^{-1}DP = A$. Hint: You know how D in $P^{-1}DP$ must look like.
- Find an eigenvector.

Bonus knowledge: Shear matrices have an eigenspace of dimensionality $d - 1$. In the above case the set of all eigenvectors must be $cv, c \in \mathbb{R}$ for some vector v.

Task 4

Show that

$$A = \begin{bmatrix} 2 & -4 \\ 13/4 & -4 \end{bmatrix}$$

- has no real eigenvalue.
- Bonus: What are its complex-valued eigenvalues??

Task 5

Bonus matrix:

- Gets its eigenvalues and eigenvectors.

Note: this is a symmetric one, so you can expect 2 eigenvalues and orthogonal eigenspaces.

$$A = \begin{bmatrix} -2 & \sqrt{24} \\ \sqrt{24} & 8 \end{bmatrix}$$

Task 6

- use numpy to get its eigenvalues and eigenvectors
- solve $Ax = (3, 17, 1/3.)$ using numpy
- (optional !!!) I would not ask this in an exam ... you will see why :'-) .

  Compute the characteristic polynomial for

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & -2 & 1 \\ 3 & 1 & 3 \end{bmatrix}$$

Task 7 (extra)

Because some of you had troubles with it
What is the cosine of the angle between

$$(6, -6, -4, \sqrt{12}),\ (6, 4, 2, \sqrt{25})\ ?$$

Task 8 (extra)

another 3x3 affine system

- show the intermediate result when the first column is the one hot vector $[1, 0, 0]$ for the first time
- show the intermediate result when the matrix has row echelon form for the first time
- get the solution

$$\begin{aligned} 2x - 3y + 2z &= -4 \\ 7x + 4.5y - 1z &= 16 \\ 4x + 3y + z &= 2 \end{aligned}$$