Linear Algebra L6 - Exam prep

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1 Learning Goals

- Recap for the exam and for test 2

Task 1

Remove vector $u = (-1, 3, -4, 2)$ from vector $v = (-2, 2, 2.5, 6)$

Task 2

Note the difference in the order between remove from and project onto.

Project vector $u = (-1, -3, -4, 2)$ onto vector $v = (3, -3, -1, 1)$

Task 3

$$\begin{aligned} x + 4y + 2z &= 5.5 \\ -5x - 22y - 5z &= -45.5 \\ 2x + 4z + 14z &= -25 \end{aligned}$$

- Show as an intermediate step the augmented matrix when for the first time the zero-th column A[:, 0] became a one-hot vector after performing transformations .
- Show as an intermediate step the augmented matrix when for the first time the augmented matrix is in row echelon form.
- Write the set of all solutions as a single vector like this,

  $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix}$$

  if there is only one solution, or an affine equation, if there is more than one solution, like this

  $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix} + s \begin{pmatrix} v_0 \\ v_1 \\ v_2 \end{pmatrix}$$

  or like this

  $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix} + s \begin{pmatrix} v_0 \\ v_1 \\ v_2 \end{pmatrix} + t \begin{pmatrix} w_0 \\ w_1 \\ w_2 \end{pmatrix}$$

  or state None if there is no solution.

Task 4

$$\begin{aligned} x + 3y - 5z &= 2.75 \\ 3x + 12y - 13z &= -9.75 \\ -4x - 6z + 25z &= -46.25 \end{aligned}$$

- Show as an intermediate step the augmented matrix when for the first time the zero-th column A[:, 0] became a one-hot vector after performing transformations .
- Show as an intermediate step the augmented matrix when for the first time the augmented matrix is in row echelon form.
- Write the set of all solutions as a single vector like this,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix}$$

if there is only one solution, or an affine equation, if there is more than one solution, like this

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix} + s \begin{pmatrix} v_0 \\ v_1 \\ v_2 \end{pmatrix}$$

or like this

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix} + s \begin{pmatrix} v_0 \\ v_1 \\ v_2 \end{pmatrix} + t \begin{pmatrix} w_0 \\ w_1 \\ w_2 \end{pmatrix}$$

or state None if there is no solution.

Task 5

Compute the inverse of

$$\begin{aligned} A_0 &= \begin{bmatrix} 9 & -2 \\ 3 & -4 \end{bmatrix} \\ A_1 &= \begin{bmatrix} 10 & 3 \\ 8 & 4 \end{bmatrix} \end{aligned}$$

Use these inverses to solve

$$\begin{aligned} A_0 x &= \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\ A_1 x &= \begin{bmatrix} -7 \\ 4 \end{bmatrix} \end{aligned}$$

Task 6

Compute the determinant of

$$A = \begin{bmatrix} 3 & -1 & 4 \\ 5 & 2.5 & 3 \\ 1 & 8 & -6 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & -2 & 0.5 \\ 2.5 & -3 & 1 \\ 3 & 2 & 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -2 & 2 \\ 8 & 3 & -2 \\ 10 & -4.5 & 5 \end{bmatrix}$$

- Are they invertible?
- Which of them has full rank ? Which of them has lower rank and which one ?

Task 7

What is the determinant of this matrix ? Write it as a polynomial in c.
For what value c the matrix is not invertible ?

$$A = \begin{bmatrix} 6 & -3 & c \\ 5 & 2 & 2 \\ -2 & -6 & -2 \end{bmatrix}$$

Task 8

Compute and apply the Householder matrix which makes transforms the first column of to a multiple of the first one-hot vector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ for

$$A = \begin{bmatrix} 8 & 1 & 2 \\ 4 & -1 & 3 \\ -8 & 4 & 2 \end{bmatrix}$$

and for (hint: here subtracting is nicer)

$$A = \begin{bmatrix} 3 & -4 & 3 \\ \sqrt{2} & 6 & 4 \\ \sqrt{5} & 3 & 2 \end{bmatrix}$$

case 0:

$$u = \begin{bmatrix}1\\2\\2\end{bmatrix} \pm \|(1,2,2)\| \begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}-2\\2\\2\end{bmatrix}$$

$$H = I - \frac{2}{12}uu^\top = I - \frac{1}{6}\begin{bmatrix}-2\\2\\2\end{bmatrix}\begin{bmatrix}-2 & 2 & 2\end{bmatrix} = I - \frac{1}{6}\begin{bmatrix}4 & -4 & -4\\-4 & 4 & 4\\-4 & 4 & 4\end{bmatrix}$$

$$= I + \frac{2}{3}\begin{bmatrix}-1 & 1 & 1\\1 & -1 & -1\\1 & -1 & -1\end{bmatrix} = \begin{bmatrix}1/3 & 2/3 & 2/3\\2/3 & 1/3 & -2/3\\2/3 & -2/3 & 1/3\end{bmatrix}$$

check possible here: $H = H^\top, HH = I$

$$HA = \begin{bmatrix}3 & 2.67 & 1.67\\0 & -0.67 & -0.67\\0 & 2.33 & -1.67\end{bmatrix}$$

case 1:

$$u = \begin{bmatrix}1\\3\\\sqrt{6}\end{bmatrix} \pm \|(1,3,\sqrt{6})\| \begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}-3\\3\\\sqrt{6}\end{bmatrix}$$

$$H = I - \frac{2}{24}uu^\top = I - \frac{1}{12}\begin{bmatrix}-3\\3\\\sqrt{6}\end{bmatrix}\begin{bmatrix}-3 & 3 & \sqrt{6}\end{bmatrix} = I - \frac{1}{12}\begin{bmatrix}9 & -9 & -3\sqrt{6}\\-9 & 9 & 3\sqrt{6}\\-3\sqrt{6} & 3\sqrt{6} & 6\end{bmatrix}$$

$$= I + \begin{bmatrix}-0.75 & 0.75 & 1/4\sqrt{6}\\0.75 & -0.75 & -1/4\sqrt{6}\\1/4\sqrt{6} & -1/4\sqrt{6} & -0.5\end{bmatrix} = \begin{bmatrix}0.25 & 0.75 & 1/4\sqrt{6}\\0.75 & 0.25 & -1/4\sqrt{6}\\1/4\sqrt{6} & -1/4\sqrt{6} & 0.5\end{bmatrix}$$

check possible here: $H = H^\top, HH = I$

$$HA = \begin{bmatrix}4 & -1 + 3/4 + 3/4\sqrt{6} & 3/4 + 3/4 + 1/4\sqrt{6}\\0 & 3/4 * -4 + 1/4 - 3/4\sqrt{6} & 3/4 * 3 + 0.25 - 1/4\sqrt{6}\\0 & 1/4\sqrt{6} * (-4) - 1/4\sqrt{6} + 0.5 * 3 & 1/4\sqrt{6} * 3 - 1/4\sqrt{6} * 1 + 0.5 * 1\end{bmatrix}$$