INF 1004 Mathematics 2
Tutorial #8

Woon Jun Wei
2200624

February 28, 2023

Question 1

Part 1

Do this one together:

$$
\begin{aligned}
x + 2y - z &= 3 \\
2x - 3y + 2z &= 5 \\
-3x + y + 5z &= 13
\end{aligned}
$$

Part 2

Solve the following affine equation systems

$$
\begin{aligned}
3x + 4y &= 1 \\
2x + 3y &= 12
\end{aligned}
$$

$$
\begin{aligned}
3x - 2y &= 4 \\
-6x + 4y &= 7
\end{aligned}
$$

$$
\begin{aligned}
2x + y + z - 6 &= 0 \\
4y + z + x &= 5 \\
2x + z + 3y &= 7
\end{aligned}
$$

Question 1 continued on next page. . .

My Solution

Part 1

$$\begin{aligned} x + 2y - z &= 3 \\ 2x - 3y + 2z &= 5 \\ -3x + y + 5z &= 13 \end{aligned}$$

Transformed into augmented matrix form:

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 3 \\ 2 & -3 & 2 & 5 \\ -3 & 1 & 5 & 13 \end{array}\right]$$

$3\rho_1 + \rho_3$: $\left[\begin{array}{rrr|r} 1 & 2 & -1 & 3 \\ 2 & -3 & 2 & 5 \\ 0 & 7 & 2 & 22 \end{array}\right]$

$2\rho_1 - \rho_2$: $\left[\begin{array}{rrr|r} 1 & 2 & -1 & 3 \\ 0 & 7 & -4 & 1 \\ 0 & 7 & 2 & 22 \end{array}\right]$

$\rho_2 - \rho_3$: $\left[\begin{array}{rrr|r} 1 & 2 & -1 & 3 \\ 0 & 7 & -4 & 1 \\ 0 & 0 & -6 & -21 \end{array}\right]$

$\frac{1}{6}\rho_3$: $\left[\begin{array}{rrr|r} 1 & 2 & -1 & 3 \\ 0 & 7 & -4 & 1 \\ 0 & 0 & -1 & \frac{21}{6} \end{array}\right]$

$\frac{1}{7}(4\rho_3 + \rho_2)$: $\left[\begin{array}{rrr|r} 1 & 2 & -1 & 3 \\ 0 & 1 & 0 & \frac{1}{7}(1 + 4 \cdot \frac{21}{6}) = \frac{15}{7} \\ 0 & 0 & 1 & \frac{21}{6} \end{array}\right]$

$\rho_1 + \rho_3 - 2\rho_2$: $\left[\begin{array}{rrr|r} 1 & 0 & 0 & 3 + \frac{21}{6} - 2 \cdot \frac{15}{7} = \frac{31}{14} \\ 0 & 1 & 0 & \frac{15}{7} \\ 0 & 0 & 1 & \frac{21}{6} \end{array}\right]$

Answer:

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & \frac{31}{14} \\ 0 & 1 & 0 & \frac{15}{7} \\ 0 & 0 & 1 & \frac{21}{6} \end{array}\right]$$

Part 2

$$\begin{aligned} 3x + 4y &= 1 \\ 2x + 3y &= 12 \end{aligned}$$

Transformed into augmented matrix form:

$$\left[\begin{array}{rr|r} 3 & 4 & 1 \\ 2 & 3 & 12 \end{array}\right]$$

$\rho_1 - \rho_2$: $\left[\begin{array}{rr|r} 3 & 4 & 1 \\ 1 & 1 & -11 \end{array}\right]$

$4\rho_2 - \rho_1$: $\left[\begin{array}{rr|r} 1 & 0 & -45 \\ 1 & 1 & -11 \end{array}\right]$

$-\rho_1 + \rho_2$: $\left[\begin{array}{rr|r} 1 & 0 & 45 \\ 0 & 1 & 34 \end{array}\right]$

Answer:

$$\left[\begin{array}{rr|r} 1 & 0 & 45 \\ 0 & 1 & 34 \end{array}\right]$$

Part 3

$$\begin{aligned} 3x - 2y &= 4 \\ -6x + 4y &= 7 \end{aligned}$$

Transformed into augmented matrix form:

$$\left[\begin{array}{rr|r} 3 & -2 & 4 \\ -6 & 4 & 7 \end{array}\right]$$

$2\rho_1 + \rho_2$: $\left[\begin{array}{rr|r} 3 & -2 & 4 \\ 0 & 0 & 15 \end{array}\right]$

This equation does not have a solution.

Question 1 continued on next page...

Part 4

$$\begin{aligned} 2x+y+z&=6\\ x+4y+z&=5\\ 3x+2y+5z&=7 \end{aligned}$$

Transformed into augmented matrix form:

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 6 \\ 1 & 4 & 1 & 5 \\ 3 & 2 & 5 & 7 \end{array}\right]$$

$\rho_1 - \rho_3$: $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 6 \\ 1 & 4 & 1 & 5 \\ 0 & -2 & 0 & -1 \end{array}\right]$

Swap ρ_3 and ρ_2, $-\frac{1}{2}\rho_2$: $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 6 \\ 0 & 1 & 0 & \frac{1}{2} \\ 1 & 4 & 1 & 5 \end{array}\right]$

$-4\rho_2 + \rho_3$: $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 6 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 3 \end{array}\right]$

$\rho_3 - \rho_1$: $\left[\begin{array}{ccc|c} -1 & -1 & 0 & -3 \\ 0 & 1 & 0 & \frac{1}{2} \\ 1 & 0 & 1 & 3 \end{array}\right]$

$-(\rho_2 + \rho_1)$: $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2.5 \\ 0 & 1 & 0 & 0.5 \\ 1 & 0 & 1 & 3 \end{array}\right]$

$\rho_1 - \rho_3$: $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2.5 \\ 0 & 1 & 0 & 0.5 \\ 0 & 0 & 1 & 0.5 \end{array}\right]$

Answer:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2.5 \\ 0 & 1 & 0 & 0.5 \\ 0 & 0 & 1 & 0.5 \end{array}\right]$$

Question 2

Modelling Problem: Solve using Gauss-Jordan elimination

- are there linear relationships?
- If so, understanding and deriving constants

In 2010, the average salary for all accountants together in the two cities San Diego, California, and Salt Lake City, Utah, was $45091.50.
The average salary in San Diego alone, however, was $5231 greater than the average salary in Salt Lake City alone. What is the average salary of an accountant in each city, assuming that there are the same number of accountants in each city?

My Solution

$$x + y = 2 \cdot 45091.50$$
$$x + y = 90183$$
$$x - y = 5231$$

In augmented matrix form:

$$\left[\begin{array}{rr|r} 1 & 1 & 90183 \\ 1 & -1 & 5231 \end{array}\right]$$

$\rho_2 + \rho_1$: $\left[\begin{array}{rr|r} 2 & 0 & 95414 \\ 1 & -1 & 90183 \end{array}\right]$

$\rho_1 - \rho_2$: $\left[\begin{array}{rr|r} 1 & 0 & 47707 \\ 1 & -1 & 5231 \end{array}\right]$

$\rho_1 - \rho_2$: $\left[\begin{array}{rr|r} 1 & 0 & 47707 \\ 0 & 1 & 42476 \end{array}\right]$

Answer:

$$\left[\begin{array}{rr|r} 1 & 0 & 47707 \\ 0 & 1 & 42476 \end{array}\right]$$

Question 3

Modelling Problem: Solve using Gauss-Jordan elimination. A chemist has prepared two acid solutions, one of which is 2% by volume, the other 7% by volume. How many cubic centimetres of each should the chemist mix together to obtain 40cm3 of a 3.2% acid solution?
Hint: If we multiply acidity per volume with a certain volume, we get a total amount of acid in this volume. If we sum 2 total amounts, we get another total amount of acid - which is the total amount for the union of the two volumes.
In order to get back to an acidity per volume, we have to divide by the volume.

My Solution

Let $\begin{cases} x = \text{volume of 2\% solution in } cm^3 \\ y = \text{volume of 7\% solution in } cm^3 \end{cases}$

$$x + y = 40$$
$$\frac{2x + 7y}{40} = 3.2$$
$$2x + 7y = 128$$

In augmented matrix form:

$$\left[\begin{array}{cc|c} 1 & 1 & 40 \\ 2 & 7 & 128 \end{array}\right]$$

$\rho_2 - 2\rho_1$: $\left[\begin{array}{cc|c} 0 & 5 & 48 \\ 2 & 7 & 128 \end{array}\right]$

$\rho_2 - \rho_1$: $\left[\begin{array}{cc|c} 0 & 5 & 48 \\ 2 & 2 & 80 \end{array}\right]$

$\frac{1}{2}\rho_2, \frac{1}{5}\rho_1$: $\left[\begin{array}{cc|c} 0 & 1 & \frac{48}{5} \\ 1 & 1 & 40 \end{array}\right]$

$\rho_2 - \rho_1$: $\left[\begin{array}{cc|c} 0 & 1 & \frac{48}{5} \\ 1 & 0 & 30.4 \end{array}\right]$

Answer:

$$\left[\begin{array}{cc|c} 0 & 1 & \frac{48}{5} \\ 1 & 0 & 30.4 \end{array}\right]$$

Question 4

In a hack and slay game, you need bags of three items which you can use to increase your attack, defence and dexterity points. The counts of each item are x, y and z respectively. The contributions of each item are shown below:

Item	Attack	Defence	Dexterity
Aunties Old Table Cloth (x)	-20	40	10
Rusty old looking dagger (y)	50	10	-10
Geylang Gift Shop Crystal (z)	10	10	60

In order to clear a final boss, you need to have 320 attack and 280 defense stats. Note that the stats scale linearly on the item equipped. Also you have 16 slots, which allow you to equip 16 items in total.
PS: Isnt it weird how a small monster can drop a huge item on death ?

1. Derive an affine equation. - Let us check in to your progress after 5 mins
2. and solve it

My Solution

$$\begin{aligned} -20x + 50y + 10z &= 320 \\ 40x + 10y + 10z &= 280 \\ x + y + z &= 16 \end{aligned}$$

In augmented matrix form:

$$\left[\begin{array}{ccc|c} -20 & 50 & 10 & 320 \\ 40 & 10 & 10 & 280 \\ 1 & 1 & 1 & 16 \end{array}\right]$$

$$\tfrac{1}{10}\rho_1, \tfrac{1}{10}\rho_2: \left[\begin{array}{ccc|c} -2 & 5 & 1 & 32 \\ 4 & 1 & 1 & 28 \\ 1 & 1 & 1 & 16 \end{array}\right]$$

$$-4\rho_3 + \rho_2: \left[\begin{array}{ccc|c} -2 & 5 & 1 & 32 \\ 0 & -3 & -3 & -36 \\ 1 & 1 & 1 & 16 \end{array}\right]$$

$$\tfrac{1}{3}\rho_2: \left[\begin{array}{ccc|c} -2 & 5 & 1 & 32 \\ 0 & -1 & -1 & -12 \\ 1 & 1 & 1 & 16 \end{array}\right]$$

$$\rho_3 + \rho_2: \left[\begin{array}{ccc|c} -2 & 5 & 1 & 32 \\ 0 & -1 & -1 & -12 \\ 1 & 0 & 0 & 4 \end{array}\right]$$

$$\rho_2 + \rho_1: \left[\begin{array}{ccc|c} -2 & 4 & 0 & 20 \\ 0 & -1 & -1 & -12 \\ 1 & 0 & 0 & 4 \end{array}\right]$$

$$2\rho_3 + \rho_1: \left[\begin{array}{ccc|c} 0 & 4 & 0 & 28 \\ 0 & -1 & -1 & -12 \\ 1 & 0 & 0 & 4 \end{array}\right]$$

$$\tfrac{1}{4}\rho_1, -\rho_2: \left[\begin{array}{ccc|c} 0 & 1 & 0 & 7 \\ 0 & 1 & 1 & 12 \\ 1 & 0 & 0 & 4 \end{array}\right]$$

$$0_2 - \rho_1: \left[\begin{array}{ccc|c} 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 5 \\ 1 & 0 & 0 & 4 \end{array}\right]$$

Answer:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

Question 5

Solve the following affine equation systems. Follow these steps:

1. Write down the augmented matrix [A—b] of the equation system above
2. Compute the reduced row echelon form.
   - Show as an intermediate step the augmented matrix when for the first time the zero-th column A[:, 0] became a one-hot vector after performing transformations .
   - Show as an intermediate step the augmented matrix when for the first time the augmented matrix is in row echelon form.
   - Show as final answer the augmented matrix in reduced row echelon form.
3. Provide one solution which solves the equation system.
4. Write the set of all solutions as a single vector like this, if there is only one solution

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix}$$

or an affine equation, if there is more than one solution, like this

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} + s \begin{bmatrix} v_0 \\ v_1 \\ v_2 \end{bmatrix}$$

or like this

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} + s \begin{bmatrix} v_0 \\ v_1 \\ v_2 \end{bmatrix} + t \begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix}$$

or state that there is no solution, if it has no solution

$$\begin{aligned} x + y + z &= 1 \\ 2x - y + z &= -1 \\ x + 3y - z &= 7 \end{aligned}$$

$$\begin{aligned} 3x - 4y &= 8 \\ x + y + z &= 2 \\ 2x - 5y - z &= 6 \end{aligned}$$

Question 5 continued on next page. . .

My Solution

Part 1

$$\begin{aligned} x+y+z&=1\\ 2x-y+z&=-1\\ x+3y-z&=7 \end{aligned}$$

In augmented matrix form:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1\\ 2 & -1 & 1 & -1\\ 1 & 3 & -1 & 7 \end{array}\right]$$

$\rho_1-\rho_3$: $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1\\ 2 & -1 & 1 & -1\\ 0 & -2 & 2 & -6 \end{array}\right]$

$\frac{1}{2}\rho_3$: $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1\\ 2 & -1 & 1 & -1\\ 0 & -1 & 1 & -3 \end{array}\right]$

$2\rho_1-\rho_2$: $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1\\ 0 & 3 & 1 & 3\\ 0 & -1 & 1 & -3 \end{array}\right]$

$\rho_2+3\rho_3$: $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1\\ 0 & 3 & 1 & 3\\ 0 & 0 & 4 & -6 \end{array}\right]$

$\frac{1}{4}\rho_3$: $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1\\ 0 & 3 & 1 & 3\\ 0 & 0 & 1 & -\frac{3}{2} \end{array}\right]$

$\frac{1}{3}(\rho_2-\rho_3)$: $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1\\ 0 & 1 & 0 & \frac{1}{3}(3-\frac{3}{2})\\ 0 & 0 & 1 & -\frac{3}{2} \end{array}\right]$

$\rho_1-\rho_2-\rho_3$: $\left[\begin{array}{ccc|r} 1 & 0 & 0 & 1-\frac{3}{2}-(-\frac{3}{2})=1\\ 0 & 1 & 0 & \frac{3}{2}\\ 0 & 0 & 1 & -\frac{3}{2} \end{array}\right]$

Answer:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1\\ 0 & 1 & 0 & \frac{3}{2}\\ 0 & 0 & 1 & -\frac{3}{2} \end{array}\right]$$

Part 2

In augmented matrix form:

$$\left[\begin{array}{ccc|c} 3 & -4 & 0 & 8\\ 1 & 1 & 1 & 2\\ 2 & -5 & -1 & 6 \end{array}\right]$$

$2\rho_2-\rho_3$: $\left[\begin{array}{ccc|c} 3 & -4 & 0 & 8\\ 1 & 1 & 1 & 2\\ 0 & 7 & 3 & -2 \end{array}\right]$

Swap ρ_2 and ρ_1: $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2\\ 3 & -4 & 0 & 8\\ 0 & 7 & 3 & -2 \end{array}\right]$

$3\rho_1-\rho_2$: $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2\\ 0 & 7 & 3 & 2\\ 0 & 7 & 3 & -2 \end{array}\right]$

$\rho_2-\rho_3$: $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2\\ 0 & 7 & 3 & 2\\ 0 & 0 & 0 & 4 \end{array}\right]$

There are no solutions to these equations

Question 6

Solve this affine equation systems.
Again, write the set of all solutions as a single vector like this, if there is only on solution

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix}$$

or an affine equation, if there is more than one solution, like this

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} + s \begin{bmatrix} v_0 \\ v_1 \\ v_2 \end{bmatrix}$$

or like this

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} + s \begin{bmatrix} v_0 \\ v_1 \\ v_2 \end{bmatrix} + t \begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix}$$

or state that there is no solution, if it has no solution

$$\begin{aligned} 2x_0 - 9x_1 - 6x_2 + 2x_3 &= 5 \\ -2x_0 + 3x_1 + 4x_2 - 2x_3 &= -2 \end{aligned}$$

$$\begin{aligned} 2x_0 - 6x_1 - 6x_2 + 3x_3 &= 5 \\ -x_0 - 2x_1 + 3x_2 - 2x_3 &= -2 \\ 2x_0 + 4x_1 - 6x_2 + 4x_3 &= 7 \end{aligned}$$

My Solution

Part 1

$$\begin{aligned} 2x_0 - 9x_1 - 6x_2 + 2x_3 &= 5 \\ -2x_0 + 3x_1 + 4x_2 - 2x_3 &= -2 \end{aligned}$$

In augmented matrix form:

$$\left[\begin{array}{rrrr|r} 2 & -9 & -6 & 2 & 5 \\ -2 & 3 & 4 & -2 & -2 \end{array}\right]$$

$\rho_1 + \rho_2$: $\left[\begin{array}{rrrr|r} 2 & -9 & -6 & 2 & 5 \\ 0 & -6 & -2 & 0 & 3 \end{array}\right]$

$3\rho_2 + \rho_1$: $\left[\begin{array}{rrrr|r} 2 & 9 & 0 & 2 & -4 \\ 0 & -6 & -2 & 0 & 3 \end{array}\right]$

There will be more than one solution, because the last row is not zero.

Part 2

$$\begin{aligned} 2x_0 - 6x_1 - 6x_2 + 3x_3 &= 5 \\ -x_0 - 2x_1 + 3x_2 - 2x_3 &= -2 \\ 2x_0 + 4x_1 - 6x_2 + 4x_3 &= 7 \end{aligned}$$

In augmented matrix form:

$$\left[\begin{array}{rrrr|r} 2 & -6 & -6 & 3 & 5 \\ -1 & -2 & 3 & -2 & -2 \\ 2 & 4 & -6 & 4 & 7 \end{array}\right]$$

$2\rho_2 + \rho_3$: $\left[\begin{array}{rrrr|r} 2 & -6 & -6 & 3 & 5 \\ -1 & -2 & 3 & -2 & -2 \\ 0 & 0 & 0 & 0 & 3 \end{array}\right]$

This equation has no solution, because the last row is zero.

Question 7

Define 2 or 3 equations in 3 variables with bias terms of your own choosing and solve it!
Verify your obtained solution x by checking that it satisfies $Ax = b$.
Do you need a Prof to write such things down?

My Solution