INF 1004 Mathematics 2
Tutorial #7

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Question 1

Compute the euclidean vector norm for vectors

$$\left[1,0,2\right],\left[3,4\right],\left[-7,2,-4,\sqrt{12}\right]$$

My Solution

Part 1

$$\|[1,0,2]\|_2 = \sqrt{1^2+0^2+2^2} = \boxed{\sqrt{5}}$$

Part 2

$$\|[3,4]\|_2 = \sqrt{3^2+4^2} = \sqrt{25} = \boxed{5}$$

Part 3

$$\begin{aligned}\|[-7,2,-4,\sqrt{12}]\|_2 &= \sqrt{(-7)^2+2^2+(-4)^2+\sqrt{12}^2} \\ &= \sqrt{49+4+16+12} \\ &= \sqrt{81} \\ &= \boxed{9}\end{aligned}$$

Question 2

Compute the corresponding unit length vector for these:

$$\left[3,4\right],\left[-1,-2,3\right],\left[-7,2,-4,\sqrt{12}\right]$$

My Solution

$$\|v\|_2 = 1$$
$$v \neq 0 \implies \frac{v}{\|v\|_2}$$

Part 1

$$\begin{aligned}\frac{[3,4]}{\|[3,4]\|_2} &= \frac{[3,4]}{5}\\ &= \frac{1}{5}\left[3,4\right]\\ &= \boxed{\left[\tfrac{3}{5},\tfrac{4}{5}\right]}\end{aligned}$$

Part 2

$$\begin{aligned}\frac{[-1,-2,3]}{\|[-1,-2,3]\|_2} &= \frac{[-1,-2,3]}{\sqrt{1^2+4^2+9^2}}\\ &= \frac{1}{\sqrt{14}}\left[-1,-2,3\right]\\ &= \boxed{\left[-\tfrac{1}{\sqrt{14}},-\tfrac{2}{\sqrt{14}},\tfrac{3}{\sqrt{14}}\right]}\end{aligned}$$

Part 3

$$\begin{aligned}\frac{[-7,2,-4,\sqrt{12}]}{\|[-7,2,-4,\sqrt{12}]\|_2} &= \frac{[-7,2,-4,\sqrt{12}]}{\sqrt{(-7)^2+2^2+(-4)^2+\sqrt{12}^2}}\\ &= \frac{1}{\sqrt{81}}\left[-7,2,-4,\sqrt{12}\right]\\ &= \frac{1}{9}\left[-7,2,-4,\sqrt{12}\right]\\ &= \boxed{\left[-\tfrac{7}{9},\tfrac{2}{9},-\tfrac{4}{9},\tfrac{\sqrt{12}}{9}\right]}\end{aligned}$$

Question 3

Compute the inner product between these vectors and their angle in degrees:

$$\begin{bmatrix}3, -2, 2\end{bmatrix}, \begin{bmatrix}1, 2, 2\end{bmatrix}$$

Compute the inner product between these vectors and their angle in degrees:

$$\begin{bmatrix}1, 0, 1\end{bmatrix}, \begin{bmatrix}2, 1, -2\end{bmatrix}, \begin{bmatrix}\frac{1}{2\sqrt{2}}, -\frac{\sqrt{3}}{2}, \frac{1}{2\sqrt{2}}\end{bmatrix}$$

My Solution

$$\frac{u}{\|u\|_2} \cdot \frac{v}{\|v\|_2} = \cos(\angle(u, v))$$

Part 1

$$\begin{aligned}\langle[3, -2, 2], [1, 2, 2]\rangle &= 3 \cdot 1 + (-2) \cdot 2 + 2 \cdot 2 \\ &= 3 + (-4) + 4 \\ &= \boxed{3}\end{aligned}$$

Angle:

Let u be $\begin{bmatrix}3, -2, 2\end{bmatrix}$ and v be $\begin{bmatrix}1, 2, 2\end{bmatrix}$

$$\begin{aligned}\cos(\angle(u, v)) &= \frac{\langle u, v\rangle}{\|u\|_2 \cdot \|v\|_2} \\ &= \frac{\begin{bmatrix}3, -2, 2\end{bmatrix} \cdot \begin{bmatrix}1, 2, 2\end{bmatrix}}{\|\begin{bmatrix}3, -2, 2\end{bmatrix}\|_2 \cdot \|\begin{bmatrix}1, 2, 2\end{bmatrix}\|_2} \\ &= \frac{3}{\sqrt{17} \cdot \sqrt{9}} \\ &= \frac{3}{\sqrt{153}}\end{aligned}$$

$$\begin{aligned}\angle(u, v) &= \cos^{-1}(\frac{3}{\sqrt{153}}) \\ &= \boxed{75.96^\circ}\end{aligned}$$

Question 3 continued on next page...

Part 2

$$\begin{aligned}\langle[1,0,1],[2,1,-2]\rangle &= 1\cdot 2+0\cdot 1+1\cdot(-2)\\ &= 2+0+(-2)\\ &= \boxed{0}\end{aligned}$$

Angle:

Let u be $\big[1,0,1\big]$ and v be $\big[2,1,-2\big]$

$$\begin{aligned}\cos(\angle(u,v)) &= \frac{\langle u,v\rangle}{\|u\|_2\cdot\|v\|_2}\\ &= \frac{\big[1,0,1\big]\cdot\big[2,1,-2\big]}{\|\big[1,0,1\big]\|_2\cdot\|\big[2,1,-2\big]\|_2}\\ &= \frac{0}{\sqrt{2}\cdot\sqrt{6}}\\ &= 0\end{aligned}$$

$$\begin{aligned}\angle(u,v) &= \cos^{-1}(0)\\ &= \boxed{90^\circ}\end{aligned}$$

Part 3

$$\begin{aligned}[2,1,2],\left[\frac{1}{2\sqrt{2}},-\frac{\sqrt{3}}{2},\frac{1}{2\sqrt{2}}\right] &= \left(2\cdot\frac{1}{2\sqrt{(2)}}\right)+\left(1\cdot-\frac{\sqrt{3}}{2}\right)+\left(2\cdot\frac{1}{2\sqrt{(2)}}\right)\\ &= \frac{1}{\sqrt{2}}-\frac{\sqrt{2}}{2}+\frac{1}{\sqrt{2}}\\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

Angle:

Let u be $\big[2,1,2\big]$ and v be $\left[\frac{1}{2\sqrt{2}},-\frac{\sqrt{3}}{2},\frac{1}{2\sqrt{2}}\right]$

Question 3 continued on next page. . .

$$
\begin{aligned}
\cos(\angle(u,v)) &= \frac{\langle u, v\rangle}{\|u\|_2 \cdot \|v\|_2} \\
&= \frac{\big[2,1,2\big] \cdot \left[\frac{1}{2\sqrt{2}}, -\frac{\sqrt{3}}{2}, \frac{1}{2\sqrt{2}}\right]}{\|\big[2,1,2\big]\|_2 \cdot \|\left[\frac{1}{2\sqrt{2}}, -\frac{\sqrt{3}}{2}, \frac{1}{2\sqrt{2}}\right]\|_2} \\
&= \frac{\frac{\sqrt{2}}{2}}{\sqrt{9} \cdot \sqrt{1}} \\
&= \frac{\frac{\sqrt{2}}{2}}{\sqrt{9}} \\
&= \frac{\sqrt{2}}{2\sqrt{9}} \\
&= \cos^{-1}(\frac{\sqrt{2}}{2\sqrt{9}}) \\
&= \boxed{76.36^\circ}
\end{aligned}
$$

Question 4

- What is the projection of $\begin{bmatrix}5, 2\end{bmatrix}$ onto the subspace spanned by vector $\begin{bmatrix}1, 1\end{bmatrix}$?
- What is the projection of $\begin{bmatrix}0, 2, 1\end{bmatrix}$ onto the subspace spanned by vector $\begin{bmatrix}1, -1, -1\end{bmatrix}$?
- Project $\begin{bmatrix}5, 2\end{bmatrix}$ onto the subspace spanned by vectors $\begin{bmatrix}2, 3\end{bmatrix}$, $\begin{bmatrix}1, 1\end{bmatrix}$
- What is the projection of $\begin{bmatrix}1, -1, 1\end{bmatrix}$ onto the subspace spanned by vector $\begin{bmatrix}1, 1, 1\end{bmatrix}$ onto the subspace spanned by vectors $\begin{bmatrix}0, 0, -1\end{bmatrix}$, $\begin{bmatrix}2, 0, 1\end{bmatrix}$? Hint: this one is more tricky. Reason: $\begin{bmatrix}0, -1, -1\end{bmatrix} \cdot \begin{bmatrix}2, 0, 1\end{bmatrix} \neq 0$

My Solution

$$x_{\|}v = \frac{x \cdot v}{v \cdot v} v = \left(x \cdot \frac{v}{\|v\|_2} \right) \frac{v}{\|v\|_2}$$

Part 1

$$\frac{\begin{bmatrix}5, 2\end{bmatrix} \cdot \begin{bmatrix}1, 1\end{bmatrix}}{2} \begin{bmatrix}1, 1\end{bmatrix} = \frac{7}{2} \begin{bmatrix}1, 1\end{bmatrix}$$

Part 2

$$\begin{aligned} \frac{\begin{bmatrix}0, 2, 1\end{bmatrix} \cdot \begin{bmatrix}1, -1, -1\end{bmatrix}}{3} \begin{bmatrix}1, -1, -1\end{bmatrix} &= \frac{-3}{3} \begin{bmatrix}1, -1, -1\end{bmatrix} \\ &= \begin{bmatrix}-1, 1, 1\end{bmatrix} \end{aligned}$$

Part 3

$$\begin{aligned} \frac{\begin{bmatrix}5, 2\end{bmatrix} \cdot \begin{bmatrix}2, 3\end{bmatrix}}{13} \begin{bmatrix}2, 3\end{bmatrix} + \frac{\begin{bmatrix}5, 2\end{bmatrix} \cdot \begin{bmatrix}1, 1\end{bmatrix}}{2} \begin{bmatrix}1, 1\end{bmatrix} &= \frac{16}{13} \begin{bmatrix}2, 3\end{bmatrix} + \frac{7}{2} \begin{bmatrix}1, 1\end{bmatrix} \\ &= \left[\tfrac{32}{13}, \tfrac{48}{13}\right] + \left[\tfrac{7}{2}, \tfrac{7}{2}\right] \\ &= \left[\tfrac{32+7}{13}, \tfrac{48+7}{13}\right] \\ &= \left[\tfrac{39}{13}, \tfrac{55}{13}\right] \end{aligned}$$

Question 5

Compute these matrix multiplications:

$$\begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ -4 & -2 \end{bmatrix}$$

$$\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \end{bmatrix}$$

Question: Do you need more of them to practice? If so, you can do at home:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 0.5 & 2.5 \\ -3.5 & 1.5 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

My Solution

Part 1

$$\begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ -4 & -2 \end{bmatrix} = \begin{bmatrix} 2 \cdot (-1) + 1 \cdot (-4) & 2 \cdot 0 + 1 \cdot (-2) \\ 3 \cdot (-1) + (-2) \cdot (-4) & 3 \cdot 0 + (-2) \cdot (-2) \end{bmatrix} = \begin{bmatrix} -6 & -2 \\ 5 & 4 \end{bmatrix}$$

$$\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \end{bmatrix} = \begin{bmatrix} -3 \cdot 2 & -3 \cdot 4 & -3 \cdot (-2) \\ 2 \cdot 2 & 2 \cdot 4 & 2 \cdot (-2) \\ 1 \cdot 2 & 1 \cdot 4 & 1 \cdot (-2) \end{bmatrix} = \begin{bmatrix} -6 & -12 & 6 \\ 4 & 8 & -4 \\ 2 & 4 & -2 \end{bmatrix}$$

Part 2

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 \cdot 3 + 2 \cdot 0 & 1 \cdot 0 + 2 \cdot 1 & 1 \cdot 1 + 2 \cdot 2 \\ 2 \cdot 3 + 4 \cdot 0 & 2 \cdot 0 + 4 \cdot 1 & 2 \cdot 1 + 4 \cdot 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 4 \\ 6 & 4 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 0.5 & 2.5 \\ -3.5 & 1.5 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 0.5 \cdot 6 + 2.5 \cdot 4 \\ 0.5 \cdot 4 + 2.5 \cdot 6 \end{bmatrix} = \begin{bmatrix} 13 \\ -15 \end{bmatrix}$$

Question 6

- Project $\begin{bmatrix}5, 2\end{bmatrix}$ onto the orthogonal space of vector $\begin{bmatrix}2, -3\end{bmatrix}$
- Project $\begin{bmatrix}1, -1, 3\end{bmatrix}$ onto the orthogonal space of vector $\begin{bmatrix}-3, 1, 1\end{bmatrix}$
- Project $\begin{bmatrix}1, -1, 3, 1\end{bmatrix}$ onto the orthogonal space of vector $\begin{bmatrix}-2, 2, 0, 0\end{bmatrix}$, $\begin{bmatrix}0, 0, \sqrt{2}, \sqrt{2}\end{bmatrix}$

My Solution

Part 1

$$\begin{aligned}\begin{bmatrix}5, 2\end{bmatrix} \cdot \begin{bmatrix}2, -3\end{bmatrix} \cdot \frac{\begin{bmatrix}2, -3\end{bmatrix}}{\|\begin{bmatrix}2, -3\end{bmatrix}\|^2} &= \frac{2 \cdot 5 + (-3) \cdot 2}{\|\begin{bmatrix}2, -3\end{bmatrix}\|^2} \\ &= \frac{13}{13} \\ &= 1\end{aligned}$$

Part 2

$$\begin{aligned}\begin{bmatrix}1, -1, 3\end{bmatrix} \cdot \begin{bmatrix}-3, 1, 1\end{bmatrix} \cdot \frac{\begin{bmatrix}-3, 1, 1\end{bmatrix}}{\|\begin{bmatrix}-3, 1, 1\end{bmatrix}\|^2} &= \frac{(-3) \cdot 1 + 1 \cdot (-1) + 1 \cdot 3}{\|\begin{bmatrix}-3, 1, 1\end{bmatrix}\|^2} \\ &= \frac{1}{\sqrt{13}} \\ &= \frac{1}{\sqrt{13}}\end{aligned}$$

Part 3

$$\begin{aligned}\begin{bmatrix}1, -1, 3, 1\end{bmatrix} \cdot \begin{bmatrix}-2, 2, 0, 0\end{bmatrix} \cdot \frac{\begin{bmatrix}-2, 2, 0, 0\end{bmatrix}}{\|\begin{bmatrix}-2, 2, 0, 0\end{bmatrix}\|^2} &= \frac{(-2) \cdot 1 + 2 \cdot (-1) + 0 \cdot 3 + 0 \cdot 1}{\|\begin{bmatrix}-2, 2, 0, 0\end{bmatrix}\|^2} \\ &= \frac{1}{\sqrt{8}} \\ &= \frac{1}{\sqrt{8}}\end{aligned}$$

$$\begin{aligned}\begin{bmatrix}1, -1, 3, 1\end{bmatrix} \cdot \begin{bmatrix}0, 0, \sqrt{2}, \sqrt{2}\end{bmatrix} \cdot \frac{\begin{bmatrix}0, 0, \sqrt{2}, \sqrt{2}\end{bmatrix}}{\|\begin{bmatrix}0, 0, \sqrt{2}, \sqrt{2}\end{bmatrix}\|^2} &= \frac{0 \cdot 1 + 0 \cdot (-1) + \sqrt{2} \cdot 3 + \sqrt{2} \cdot 1}{\|\begin{bmatrix}0, 0, \sqrt{2}, \sqrt{2}\end{bmatrix}\|^2} \\ &= \frac{3 + \sqrt{2}}{\sqrt{2}} \\ &= \frac{3 + \sqrt{2}}{\sqrt{2}}\end{aligned}$$

Question 7

Run Gram-Schmid-orthogonalization on the vectors

$$[12, 12, 6], [2, -2, 4], [-2, -2, 1]$$

My Solution

Part 1

$$\begin{aligned}
[12, 12, 6] &= [12, 12, 6] \\
[2, -2, 4] &= [2, -2, 4] - \frac{2 \cdot 12 + (-2) \cdot 12 + 4 \cdot 6}{\|[12, 12, 6]\|^2} \cdot [12, 12, 6] \\
&= [2, -2, 4] - \frac{144}{144} \cdot [12, 12, 6] \\
&= [2, -2, 4] - [12, 12, 6] \\
&= [-10, -14, -2] \\
[-2, -2, 1] &= [-2, -2, 1] - \frac{(-2) \cdot 12 + (-2) \cdot 12 + 1 \cdot 6}{\|[12, 12, 6]\|^2} \cdot [12, 12, 6] \\
&= [-2, -2, 1] - \frac{0}{144} \cdot [12, 12, 6] \\
&= [-2, -2, 1] - [0, 0, 0] \\
&= [-2, -2, 1]
\end{aligned}$$

Question 8

Understanding Distances coming from ℓ_p-norms

Coding: plot in python or similar the set of points $x \in \mathbb{R}^2$ usuch that $\|x\|_p = 1$ for

- $p = 0.2$
- $p = 0.5$
- $p = 1$
- $p = 1.5$
- $p = 2$
- $p = 4$
- $p = 8$
- $p = 16$

Hint: in 2 dimensions for $p = 2$ the solution is given by

$$x(t) = (\cos(t), \sin(t))$$

due to $\cos^2(t) + \sin^2(t) = 1$.
you can use the same idea with different powers. You can start by considering $(\cos^r(t), \sin^r(t))$. One thing to node: $\cos(t)^r + \sin(t)^r$ is not always defined for negative values and certain r.

For $p \neq 2$, you can consider this, thich deals with the signs:

$$x(t) = (sign(\cos(t))|\cos(t)|^r, sign(\sin(t))|\sin(t)|^r)$$

for the right choice of r. Find out which r is suitable for a general $p > 0$ such that $\|x\|_p = 1$. Then plot it in python.

My Solution

Part 1

$$\begin{aligned}
\left[1, -1, 3, 1\right] \cdot \left[1, 1, 1, 1\right] \cdot \frac{\left[1, 1, 1, 1\right]}{\|\left[1, 1, 1, 1\right]\|^2} &= \frac{1 \cdot 1 + (-1) \cdot 1 + 3 \cdot 1 + 1 \cdot 1}{\|\left[1, 1, 1, 1\right]\|^2} \\
&= \frac{4}{2} \\
&= 2
\end{aligned}$$