Linear Algebra L2 - Affine equation systems

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Learning Goals

- Solving affine equation systems

Task 1

Do this one together:

$$\begin{aligned} x + 2y - z &= 3 \\ 2x - 3y + 2z &= 5 \\ -3x + y + 5z &= 13 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & -1 & \vert & 3 \\ 2 & -3 & 2 & \vert & 5 \\ -3 & 1 & 5 & \vert & 13 \end{bmatrix} \to r_1 + = -2r_0, r_2 + = 3r_0, : \begin{bmatrix} 1 & 2 & -1 & \vert & 3 \\ 0 & -7 & 4 & \vert & -1 \\ 0 & 7 & 2 & \vert & 22 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & \vert & 3 \\ 0 & -7 & 4 & \vert & -1 \\ 0 & 0 & 6 & \vert & 21 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & -1 & \vert & 3 \\ 0 & -7 & 4 & \vert & -1 \\ 0 & 0 & 1 & \vert & 3.5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & \vert & 3 \\ 0 & 1 & -4/7 & \vert & 1/7 \\ 0 & 0 & 1 & \vert & 3.5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & \vert & 6.5 \\ 0 & 1 & 0 & \vert & 2 + 1/7 \\ 0 & 0 & 1 & \vert & 3.5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & \vert & 6.5 - 2 * 2 - 2/7 \\ 0 & 1 & 0 & \vert & 2 + 1/7 \\ 0 & 0 & 1 & \vert & 3.5 \end{bmatrix}$$

$$x = 2.5 - 2/7, y = 2 + 1/7, z = 3.5$$

Verify!

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & -3 & 2 \\ -3 & 1 & 5 \end{bmatrix} \begin{bmatrix} 2.5 - 2/7 \\ 2 + 1/7 \\ 3.5 \end{bmatrix} = \begin{bmatrix} 2.5 - 2/7 + 4 + 2/7 - 3.5 \\ 5 - 4/7 - 6 - 3/7 + 7 \\ -7.5 + 6/7 + 2 + 1/7 + 17.5 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 13 \end{bmatrix}$$

Solve the following affine equation systems

$$\begin{aligned} 3x + 4y &= 1 \\ 2x + 3y &= 12 \end{aligned}$$

$$\begin{bmatrix} 3 & 4 & \vert & 1 \\ 2 & 3 & \vert & 12 \end{bmatrix} \to \begin{bmatrix} 1 & 4/3 & \vert & 1/3 \\ 2 & 3 & \vert & 12 \end{bmatrix} \to (-2) \begin{bmatrix} 1 & 4/3 & \vert & 1/3 \\ 0 & 3 - 8/3 & \vert & 12 - 2/3 \end{bmatrix} \to \begin{bmatrix} 1 & 4/3 & \vert & 1/3 \\ 0 & 1/3 & \vert & 12 - 2/3 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 4/3 & \vert & 1/3 \\ 0 & 1 & \vert & 34 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \vert & 1/3 + 34 * (-4/3) \\ 0 & 1 & \vert & 34 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \vert & 1/3 - 34 + 34 * (-1/3) = 1/3 - 34 - 11 - 1/3 \\ 0 & 1 & \vert & 34 \end{bmatrix}$$

Verify!

$$\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -45 \\ 34 \end{bmatrix} = \begin{bmatrix} 3 * (-45) + 4 * 34 \\ 2 * (-45) + 3 * 34 \end{bmatrix} = \begin{bmatrix} -135 + 136 \\ -90 + 102 \end{bmatrix} = \begin{bmatrix} 1 \\ 12 \end{bmatrix}$$

$$\begin{aligned} 3x - 2y &= 4 \\ -6x + 4y &= 7 \end{aligned}$$

$$\begin{bmatrix} 3 & -2 & | & 4 \\ -6 & 4 & | & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & -2 & | & 4 \\ 0 & 0 & | & 15 \end{bmatrix}$$

No solution! Cannot satisfy

$$0x_0 + 0x_1 = 15$$

$$\begin{aligned} 2x + y + z - 6 &= 0 \\ 4y + z + x &= 5 \\ 2x + z + 3y &= 7 \end{aligned}$$

$$\begin{bmatrix} 2 & 1 & 1 & | & 6 \\ 1 & 4 & 1 & | & 5 \\ 2 & 3 & 1 & | & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1/2 & 1/2 & | & 3 \\ 1 & 4 & 1 & | & 5 \\ 2 & 3 & 1 & | & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1/2 & 1/2 & | & 3 \\ 0 & 3.5 & 0.5 & | & 2 \\ 0 & 2 & 0 & | & 1 \end{bmatrix} \rightarrow \text{swap} r_1, r_2 \begin{bmatrix} 1 & 1/2 & 1/2 & | & 3 \\ 0 & 2 & 0 & | & 1 \\ 0 & 3.5 & 0.5 & | & 2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1/2 & 1/2 & | & 3 \\ 0 & 1 & 0 & | & 1/2 \\ 0 & 3.5 & 0.5 & | & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1/2 & 1/2 & | & 3 \\ 0 & 1 & 0 & | & 1/2 \\ 0 & 0 & 0.5 & | & 1/4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1/2 & 1/2 & | & 3 \\ 0 & 1 & 0 & | & 1/2 \\ 0 & 0 & 1 & | & 1/2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1/2 & 0 & | & 2.75 \\ 0 & 1 & 0 & | & 1/2 \\ 0 & 0 & 1 & | & 1/2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & | & 2.5 \\ 0 & 1 & 0 & | & 1/2 \\ 0 & 0 & 1 & | & 1/2 \end{bmatrix}$$

Verify again!

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2.5 \\ 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 5 + 0.5 + 0.5 \\ 2.5 + 2 + 0.5 \\ 5 + 1.5 + 0.5 \end{bmatrix}$$

Note: you can choose to swap rows, if it makes calculations easier.

Task 2

Modelling Problem: Solve using Gauss-Jordan elimination.

- are there linear relationships ?
- If so, understanding and deriving constraints

In 2010, the average salary for all accountants together in the two cities San Diego, California, and Salt Lake City, Utah, was $\$45091.50$.
The average salary in San Diego alone, however, was $\$5231$ greater than the average salary in Salt Lake City alone. What is the average salary of an accountant in each city, assuming that there are the same number of accountants in each city?

Let the average salaries per city be: s_0, s_1.

To understand: You can compute an average from averages, when you know the respective counts n_0, n_1

$$\overline{s_{\text{0 and 1}}} = (n_0 s_0 + n_1 s_1)/(n_0 + n_1)$$

Reason: Let the total counts be a_0, a_1. Then:

$$\begin{aligned} n_0 s_0 &= a_0 \\ n_1 s_1 &= a_1 \\ \Rightarrow n_0 s_0 + n_1 s_1 &= a_0 + a_1 \\ \overline{s_{\text{0 and 1}}} &= \frac{a_0 + a_1}{n_0 + n_1} \end{aligned}$$

Let the average salaries per city be: s_0, s_1, then

$$\begin{aligned} (n_{a,sd} s_0 + n_{a,sl} s_1)/(n_{a,sd} + n_{a,sl}) &= 45091.5 \\ s_0 + 5231 &= s_1 \\ n_{a,sd} &= n_{a,sl} \end{aligned}$$

Transforms using into $n_{a,sd} = n_{a,sl}$

$$\begin{aligned} n_{a,sd}(s_0 + s_1)/(n_{a,sd} + n_{a,sd}) &= 45091.5 \\ s_0 + 5231 &= s_1 \\ n_{a,sd} &= n_{a,sl} \end{aligned}$$

thus:

$$\begin{aligned} (s_0 + s_1)/2 &= 45091.5 \\ s_0 + 5231 &= s_1 \end{aligned}$$

$$\begin{aligned} \begin{bmatrix} 1/2 & 1/2 & | & 45091.5 \\ 1 & -1 & | & 5321 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & | & 90183 \\ 1 & -1 & | & 5321 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & | & 90183 \\ 0 & -2 & | & 5321 - 90183 \end{bmatrix} &\rightarrow \begin{bmatrix} 1 & 1 & | & 90183 \\ 0 & 1 & | & 42431 \end{bmatrix} \\ &\rightarrow \begin{bmatrix} 1 & 0 & | & 47752 \\ 0 & 1 & | & 42431 \end{bmatrix} \end{aligned}$$

Task 3

Modelling Problem: Solve using Gauss-Jordan elimination. A chemist has prepared two acid solutions, one of which is 2% by volume, the other 7% by volume. How many cubic centimetres of each should the chemist mix together to obtain $40cm^3$ of a 3.2% acid solution?

Hint: If we multiply acidity per volume with a certain volume, we get a total amount of acid in this volume. If we sum 2 total amounts, we get another total amount of acid - which is the total amount for the union of the two volumes.
In order to get back to an acidity per volume, we have to divide by the volume. So

$$\frac{2/100 v_0 + 7/100 v_1}{(v_1 + v_2)} = 3.2/100$$

The second constraint is:

$$v_0 + v_1 = 40$$

This transforms into:

$$\begin{aligned} 2/100 v_0 + 7/100 v_1 &= 3.2/100(v_1 + v_2) \\ v_0 + v_1 &= 40 \end{aligned}$$

and into

$$(2-3.2)v_0 + (7-3.2)v_1 = 0$$
$$v_0 + v_1 = 40$$

$$\begin{bmatrix} -1.2 & 3.8 & | & 0 \\ 1 & 1 & | & 40 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & | & 40 \\ -1.2 & 3.8 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & | & 40 \\ 0 & 5 & | & 48 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & | & 40 \\ 0 & 1 & | & 9.6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & | & 30.4 \\ 0 & 1 & | & 9.6 \end{bmatrix}$$

Task 4

In a hack and slay game, you need bags of three items which you can use to increase your attack, defence and dexterity points. The counts of each item are x, y and z respectively.
The contributions of each item are shown below:

item	Attack	Defense	Dex
Aunties Old Table Cloth (x)	-20	40	10
Rusty old looking dagger (y)	50	10	-10
Geylang Gift Shop Crystal (z)	10	10	60

In order to clear a final boss, you need to have 320 attack and 280 defense stats. Note that the stats scale linearly on the item equipped. Also you have 16 slots, which allow you to equip 16 items in total.

PS: Isnt it weird how a small monster can drop a huge item on death ?
a) Derive an affine equation. - Let us check in to your progress after 5 minutes .

$$-20a_0 + 50a_1 + 10a_2 = 320$$
$$40a_0 + 10a_1 + 10a_2 = 280$$
$$a_0 + a_1 + a_2 = 16$$

Note: in practice one would want to solve this via inequalities: Equip at most 16 items, have at least those stats. Here it is a bit idealised. Guess why the Geylang Gift Shop Crystal can make you run so fast (high Dexterity)?
b) and solve it.

$$\begin{bmatrix} -20 & 50 & 10 & | & 320 \\ 40 & 10 & 10 & | & 280 \\ 1 & 1 & 1 & | & 16 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & | & 16 \\ -2 & 5 & 1 & | & 32 \\ 4 & 1 & 1 & | & 28 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & | & 16 \\ 0 & 7 & 3 & | & 64 \\ 0 & -3 & -3 & | & -36 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & 1 & | & 16 \\ 0 & 7 & 3 & | & 64 \\ 0 & 1 & 1 & | & 12 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & | & 16 \\ 0 & 1 & 1 & | & 12 \\ 0 & 7 & 3 & | & 64 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & | & 16 \\ 0 & 1 & 1 & | & 12 \\ 0 & 0 & -4 & | & -20 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & | & 16 \\ 0 & 1 & 1 & | & 12 \\ 0 & 0 & 1 & | & 5 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & 0 & | & 11 \\ 0 & 1 & 0 & | & 7 \\ 0 & 0 & 1 & | & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 4 \\ 0 & 1 & 0 & | & 7 \\ 0 & 0 & 1 & | & 5 \end{bmatrix}$$

Next time you need to slay something, take Table Cloth with you.

Task 5

Solve the following affine equation systems. Follow these steps:
a) Write down the augmented matrix [A|b] of the equation system above
b) Compute the reduced row echelon form.

- Show as an intermediate step the augmented matrix when for the first time the zero-th column A[:, 0] became a one-hot vector after performing transformations .
- Show as an intermediate step the augmented matrix when for the first time the augmented matrix is in row echelon form.

- Show as final answer the augmented matrix in reduced row echelon form.

c) Provide one solution which solves the equation system.
d) Write the set of all solutions as a single vector like this, if there is only one solution

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix}$$

or an affine equation, if there is more than one solution, like this

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} + s \begin{bmatrix} v_0 \\ v_1 \\ v_2 \end{bmatrix}$$

or like this

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} + s \begin{bmatrix} v_0 \\ v_1 \\ v_2 \end{bmatrix} + t \begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix}$$

or state that there is no solution, if it has no solution.

For these:

$$\begin{aligned} x + y + z &= 1 \\ 2x - y + z &= -1 \\ x + 3y - z &= 7 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 1 & | & 1 \\ 2 & -1 & 1 & | & -1 \\ 1 & 3 & -1 & | & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & | & 1 \\ 0 & -3 & -1 & | & -3 \\ 0 & 2 & -2 & | & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & | & 1 \\ 0 & 1 & 1/3 & | & 1 \\ 0 & 2 & -2 & | & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & | & 1 \\ 0 & 1 & 1/3 & | & 1 \\ 0 & 0 & -2-2/3 & | & 4 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 1 & 1 & | & 1 \\ 0 & 1 & 1/3 & | & 1 \\ 0 & 0 & -8/3 & | & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & | & 1 \\ 0 & 1 & 1/3 & | & 1 \\ 0 & 0 & 1 & | & -3/2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & | & 5/2 \\ 0 & 1 & 0 & | & 3/2 \\ 0 & 0 & 1 & | & -3/2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & 3/2 \\ 0 & 0 & 1 & | & -3/2 \end{bmatrix}$$

first time one-hot vector:

$$\begin{bmatrix} 1 & 1 & 1 & | & 1 \\ 0 & -3 & -1 & | & -3 \\ 0 & 2 & -2 & | & 6 \end{bmatrix}$$

first time row echelon form:

$$\begin{bmatrix} 1 & 1 & 1 & | & 1 \\ 0 & 1 & 1/3 & | & 1 \\ 0 & 0 & 1 & | & -3/2 \end{bmatrix}$$

$$\begin{aligned} 3x - 4y &= 8 \\ x + y + z &= 2 \\ 2x - 5y - z &= 6 \end{aligned}$$

$$\begin{bmatrix} 3 & -4 & 0 & | & 8 \\ 1 & 1 & 1 & | & 2 \\ 2 & -5 & -1 & | & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & | & 2 \\ 3 & -4 & 0 & | & 8 \\ 2 & -5 & -1 & | & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & | & 2 \\ 0 & -7 & -3 & | & 2 \\ 0 & -7 & -3 & | & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & | & 2 \\ 0 & -7 & -3 & | & 2 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & 1 & | & 2 \\ 0 & 1 & 3/7 & | & -2/7 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 4/7 & | & 2+2/7 \\ 0 & 1 & 3/7 & | & -2/7 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

first time one-hot vector:

$$\begin{bmatrix} 1 & 1 & 1 & | & 2 \\ 0 & -7 & -3 & | & 2 \\ 0 & -7 & -3 & | & 2 \end{bmatrix}$$

first time row echelon form:

$$\begin{bmatrix} 1 & 1 & 1 & | & 2 \\ 0 & 1 & 3/7 & | & -2/7 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

One has a one-dim affine space of solutions here.

One solution is: $x_2 = 0$, $x_1 = -2/7$, $x_0 = 2 + 2/7 = 16/7$
But one can plug in a different value for x_2, the solutions as a function of x_2 are:

$$x_1 = -2/7 - 3/7x_2$$
$$x_0 = 16/7 - 4/7x_2$$

One can write them as affine space .
How ?
Step 1: order nicely

$$x_0 = 16/7 - 4/7x_2$$
$$x_1 = -2/7 - 3/7x_2$$

Step 2: parametrize every free variable by some variable. The only free variable is x_2. So set : $x_2 = t$.

$$x_0 = 16/7 - 4/7x_2$$
$$x_1 = -2/7 - 3/7x_2$$
$$x_2 = t$$

Now replace x_2 by the parametrization:

$$x_0 = 16/7 - 4/7t$$
$$x_1 = -2/7 - 3/7t$$
$$x_2 = t$$

Now write this as a sum of vectors:

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 16/7 \\ -2/7 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4/7 \\ -3/7 \\ 1 \end{bmatrix}$$

Task 6

Solve these affine equation systems:
Again, write the set of all solutions as a single vector like this, if there is only one solution

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix}$$

or an affine equation, if there is more than one solution, like this

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} + s \begin{bmatrix} v_0 \\ v_1 \\ v_2 \end{bmatrix}$$

or like this

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} + s \begin{bmatrix} v_0 \\ v_1 \\ v_2 \end{bmatrix} + t \begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix}$$

or state that there is no solution, if it has no solution.

$$\begin{aligned} 3x_0 - 9x_1 - 6x_2 + 2x_3 &= 5 \\ -2x_0 + 3x_1 + 4x_2 - 2x_3 &= -2 \end{aligned}$$

$$\begin{bmatrix} 3 & -9 & -6 & 2 & | & 5 \\ -2 & 3 & 4 & -2 & | & -2 \end{bmatrix}$$
$$\rightarrow \begin{bmatrix} 1 & -3 & -2 & 2/3 & | & 5/3 \\ -2 & 3 & 4 & -2 & | & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & -2 & 2/3 & | & 5/3 \\ 0 & -3 & 0 & -2+4/3 & | & -2+10/3 \end{bmatrix}$$
$$\rightarrow \begin{bmatrix} 1 & 0 & -2 & 2-2/3 & | & 2-5/3 \\ 0 & -3 & 0 & -2+4/3 & | & -2+10/3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 & 2-2/3 & | & 2-5/3 \\ 0 & 1 & 0 & 2/3-4/9 & | & 2/3-10/9 \end{bmatrix}$$

first time one-hot vector:

$$\begin{bmatrix} 1 & -3 & -2 & 2/3 & | & 5/3 \\ 0 & -3 & 0 & -2+4/3 & | & -2+10/3 \end{bmatrix}$$

first time row echelon form (and also the reduced row echelon form):

$$\begin{bmatrix} 1 & 0 & -2 & 2-2/3 & | & 2-5/3 \\ 0 & 1 & 0 & 2/3-4/9 & | & 2/3-10/9 \end{bmatrix}$$

The solution is a two-dim affine space. You can write it a as a function of those variables which have no left-most 1-entry:

$$\begin{aligned} x_1 &= 2/3 - 10/9 - 0x_2 - (2/3 - 4/9)x_3 \\ x_0 &= 2 - 5/3 + 2x_2 - (2 - 2/3)x_3 \end{aligned}$$

Reorder:

$$\begin{aligned} x_0 &= 2 - 5/3 + 2x_2 - (2 - 2/3)x_3 \\ x_1 &= 2/3 - 10/9 - 0x_2 - (2/3 - 4/9)x_3 \end{aligned}$$

parametrize $x_2 = t$, $x_3 = u$:

$$\begin{aligned}
x_0 &= 2 - 5/3 + 2x_2 - (2 - 2/3)x_3 \\
x_1 &= 2/3 - 10/9 - 0x_2 - (2/3 - 4/9)x_3 \\
x_2 &= t \\
x_3 &= u
\end{aligned}$$

replace everywhere what you just have parametrized:

$$\begin{aligned}
x_0 &= 2 - 5/3 + 2t - (2 - 2/3)u \\
x_1 &= 2/3 - 10/9 - (2/3 - 4/9)u \\
x_2 &= t \\
x_3 &= u
\end{aligned}$$

write as affine space:

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 - 5/3 \\ 2/3 - 10/9 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} -(2 - 2/3) \\ -(2/3 - 4/9) \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{aligned}
2x_0 - 6x_1 - 6x_2 + 3x_3 &= 5 \\
-x_0 - 2x_1 + 3x_2 - 2x_3 &= -2 \\
2x_0 + 4x_1 - 6x_2 + 4x_3 &= 7
\end{aligned}$$

$$\begin{bmatrix} 2 & -6 & -6 & 3 & | & 5 \\ -1 & -2 & 3 & -2 & | & -2 \\ 2 & 4 & -6 & 4 & | & 7 \end{bmatrix} \to \begin{bmatrix} 2 & -6 & -6 & 3 & | & 5 \\ 0 & -5 & 0 & -1/2 & | & 1/2 \\ 0 & 10 & 0 & 1 & | & 2 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & -3 & -3 & 1.5 & | & 2.5 \\ 0 & -5 & 0 & -1/2 & | & 1/2 \\ 0 & 10 & 0 & 1 & | & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & -3 & 1.5 & | & 2.5 \\ 0 & -5 & 0 & -1/2 & | & 1/2 \\ 0 & 0 & 0 & 0 & | & 3 \end{bmatrix}$$

first time one-hot vector:

$$\begin{bmatrix} 1 & -3 & -3 & 1.5 & | & 2.5 \\ 0 & -5 & 0 & -1/2 & | & 1/2 \\ 0 & 10 & 0 & 1 & | & 2 \end{bmatrix}$$

first time row echelon form:
You never get there in this case.

No solution! Even when one has less equations than variables

Task 7

:)
Whatever you come up, you are able to verify the solution!! (by checking whether $Ax = b$ holds for your solution x)

So you can train yourself - by giving you 3x3 matrices with biases and trying to solve them

to prepare you to defeat the math attacks coming from Profs