Linear Algebra L5 - Eigenvalues and eigenvectors

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1 Learning Goals

- Eigenvalues and Eigenvectors

Task 1

- Compute the eigenvalues and eigenvectors for the matrices below.
- For one of these matrices compute the the matrix P such that $P^{-1}DP = A$, and verify that PAP^{-1} is the diagonal matrix of the eigenvalues.

$$A = \begin{bmatrix} 4 & \sqrt{15} \\ \sqrt{15} & 2 \end{bmatrix} \text{ for this one we do it together}$$
$$A = \begin{bmatrix} 2 & -2 \\ -3 & 1 \end{bmatrix}$$
$$A = \begin{bmatrix} 2 & -4 \\ 4 & -6 \end{bmatrix}$$

a)

$$\begin{aligned} A-(-1)I &= \begin{bmatrix} 5 & \sqrt{15} \\ \sqrt{15} & 3 \end{bmatrix} \to \begin{bmatrix} 5\sqrt{15} & 15 \\ \sqrt{15} & 3 \end{bmatrix} \to \begin{bmatrix} \sqrt{15} & 3 \\ \sqrt{15} & 3 \end{bmatrix} \\ v_0 = (-3, \sqrt{15}) \\ A-(7)I &= \begin{bmatrix} -3 & \sqrt{15} \\ \sqrt{15} & -5 \end{bmatrix} \to \begin{bmatrix} -3\sqrt{15} & 15 \\ \sqrt{15} & -5 \end{bmatrix} \to \begin{bmatrix} \sqrt{15} & -5 \\ \sqrt{15} & -5 \end{bmatrix} \\ v_1 = (5, \sqrt{15}) \end{aligned}$$

case: symmetric matrix, 2 different eigenvalues,
eigenspaces are orthogonal for the diff eigenvalues check: $v_0 \cdot v_1 = -15 + \sqrt{15}\sqrt{15} = 0$
b)

$$\begin{aligned} A-(-1)I &= \begin{bmatrix} 3 & -2 \\ -3 & 2 \end{bmatrix} v = 0 \\ v_0 &= (2,3) \\ A-4I &= \begin{bmatrix} -2 & -2 \\ -3 & -3 \end{bmatrix} v = 0 \\ v_1 &= (1,-1) \end{aligned}$$

Eigenspaces are not orthogonal, but matrix is not symmetric! So it is ok.
c)

$$A - (-2I) = \begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix}$$
$$v = (1, 1)$$

Eigenspace space just one-dim $\{c(1,1), c \in \mathbb{R}\}$, but matrix has no differing eigenvalues, so ok!

Task 2

- Compute an eigenvector for the eigenvalue $x = 3$ for the below $(3,3)$-matrix. Note: nobody asks you to compute its characteristic polynomial or to get all of its eigenvalues (Prof did it).
- Validate that the found eigenvector v is indeed the correct one, that is , that $Av = 3v$ holds.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 2 & 4 & 4 \end{bmatrix}$$

$$\begin{aligned} (1-x)(1-x)(4-x) + 0(-1)2 + 0 - 1(1-x)2 - (1-x)(-1)4 \\ = (1-x)(1-x)(4-x) + 2(1-x) = 0 \\ = -x^3 + x^2(4+1+1) - x(1+4+4) + 4 - x(2) + 2 \\ 0 = x^3 - 6x^2 + 11x - 6 \end{aligned}$$

$x = 3$

$$A - 3I = \begin{bmatrix} -2 & 0 & 1 \\ 0 & -2 & -1 \\ 2 & 4 & 1 \end{bmatrix} \to \begin{bmatrix} -2 & 0 & 1 \\ 0 & -2 & -1 \\ 0 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} -2 & 0 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$
$$v_0 = (1, -1, 2)$$

Proof:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 2 & 4 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 2-4+8 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 6 \end{bmatrix}$$

Task 3

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

- Show that this matrix has only the eigenvalue 1, twice.
- Prove that there cannot exist any matrix P such that $P^{-1}DP = A$. Hint: You know how D in $P^{-1}DP$ must look like.
- Find an eigenvector.

Bonus knowledge: Shear matrices have an eigenspace of dimensionality $d - 1$. In the above case the set of all eigenvectors must be $cv, c \in \mathbb{R}$ for some vector v.

Obviously $f(\lambda) = (1 - \lambda)^2$, so only $\lambda = 1$ is a solution. It has two eigenvalues equal to 1.

Therefore $D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$. If there was a P such that $P^{-1}DP = A$, then $P^{-1}DP = P^{-1}IP = P^{-1}P = I$ and therefore $A = I$, which contradicts the fact that A is not the identity matrix.

$$A - I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

The second row poses no constraint. The first row says for $x = (x_0, x_1)$ that it must be $x_1 = 0$. Therefore It has only $v_0 = (a, 0)$ as solutions.

Task 4

Show that

$$A = \begin{bmatrix} 2 & -4 \\ 13/4 & -4 \end{bmatrix}$$

- has no real eigenvalue.

- Bonus: What are its complex-valued eigenvalues??

$$f(\lambda) = (2 - x)(-4 - x) + 9 = x^2 + 2x - 8 + 13 = x^2 + 2x + 5$$

Its solutions are $x_{0/1} = -1 \pm \sqrt{1-5} = -1 \pm 2\sqrt{-1} = -1 \pm 2i$

Task 5

Bonus matrix:

- Gets its eigenvalues and eigenvectors.

Note: this is a symmetric one, so you can expect 2 eigenvalues and orthogonal eigenspaces.

$$A = \begin{bmatrix} -2 & \sqrt{24} \\ \sqrt{24} & 8 \end{bmatrix}$$

$$\begin{aligned} A - -4I &= \begin{bmatrix} 2 & \sqrt{24} \\ \sqrt{24} & 12 \end{bmatrix} \to \begin{bmatrix} 2\sqrt{24} & 24 \\ \sqrt{24} & 12 \end{bmatrix} \\ v_0 = (-12, \sqrt{24}) & \\ A - 10I &= \begin{bmatrix} -12 & \sqrt{24} \\ \sqrt{24} & -2 \end{bmatrix} \to \begin{bmatrix} -12 & \sqrt{24} \\ 24 & -2\sqrt{24} \end{bmatrix} \\ v_1 = (\sqrt{24}, 12) & \end{aligned}$$

case: symmetric matrix, 2 different eigenvalues,
eigenspaces are orthogonal for the diff eigenvalues

Task 6

- use numpy to get its eigenvalues and eigenvectors
- solve $Ax = (3, 17, 1/3.)$ using numpy

- (optional !!!) I would not ask this in an exam ... you will see why :'-) .
  s

  Compute the characteristic polynomial for

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & -2 & 1 \\ 3 & 1 & 3 \end{bmatrix}$$

$$\begin{aligned} f(c) &= (1-x)(-2-x)(3-x) + 2*1*3 + 4*2*1 - 4(-2-x)3 - (1-x)1*1 - 2*2*(3-x) \\ &= (-2+2x-x+x^2)(3-x) + 12(x+2) + (x-1) + 4(x-3) + 14 \\ &= (-2+x+x^2)(3-x) + 12x + 24 + x - 1 + 4x - 12 + 14 \\ &= -6 + 3x + 3x^2 + 2x - x^2 - x^3 + 17x + 25 \\ &= -x^3 + 2x^2 + 22x + 19 \end{aligned}$$

Task 7 (extra)

Because some of you had troubles with it
What is the cosine of the angle between

$$(6, -6, -4, \sqrt{12}),\ (6, 4, 2, \sqrt{25})\ ?$$

Task 8 (extra)

another 3x3 affine system

- show the intermediate result when the first column is the one hot vector $[1, 0, 0]$ for the first time
- show the intermediate result when the matrix has row echelon form for the first time
- get the solution

$$\begin{aligned} 2x - 3y + 2z &= -4 \\ 7x + 4.5y - 1z &= 16 \\ 4x + 3y + z &= 2 \end{aligned}$$