INF 1004 Mathematics 2
Tutorial #4 & #5

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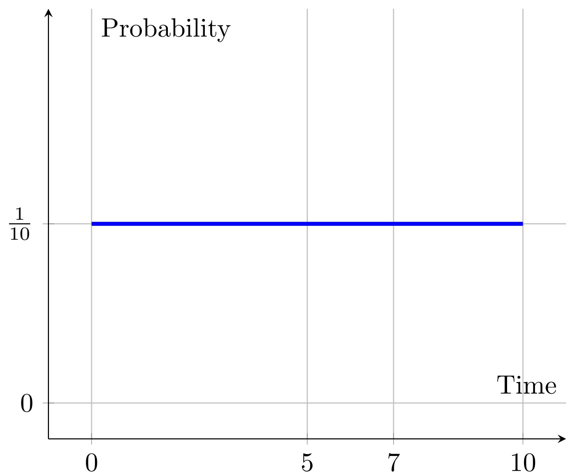
Question 1

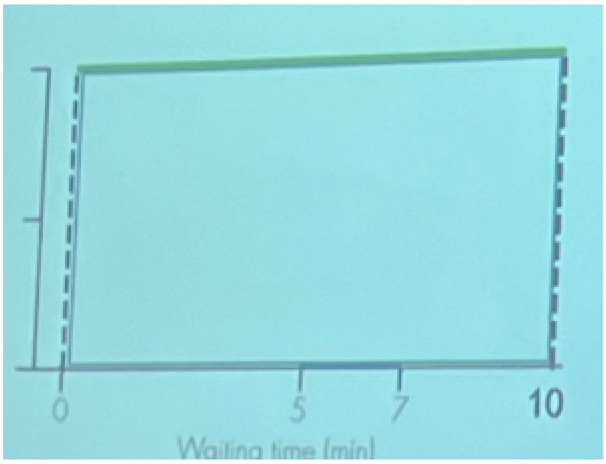
Suppose that buses arrive at a bus-stop at every 10-min interval. What is the probability that a passenger who arrives at the bus-stop will have to wait between 5-7 mins? Assume all possibilities are equally likely. Hints:

1. Is this a uniform distribution or a normal distribution?
2. What is the minimum in this distribution?
3. What is the maximum in this distribution?
4. Draw the probability distribution for this problem.

Sample Solutions

1. Uniform distribution
2. What is the minimum in this distribution? 0
3. What is the maximum in this distribution? 10
4. Draw the probability distribution for this problem.





Let X be the time that the passenger needs to wait for the bus.

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$$
\begin{aligned}
P(5 < X < 7) &= (7-5)(0.1) \\
&= 0.2
\end{aligned}
$$

Question 2

Stanford-Binet IQ Test scores are normally distributed with a mean score of 100 and a standard deviation of 16. Find the probability that a randomly selected person has an IQ test score:

(a) Over 140.

(b) Under 88.

(c) Between 72 and 128.

Sample Solutions

Given:

$\mu = 100$
$\sigma = 16$

$$X \sim N(100, 16^2)$$

(a)

$$\begin{aligned} P(X > 140) &= P(Z > \frac{140 - 100}{16}) \\ &= P(Z > 2.5) \\ &= 1 - P(Z \leq 2.5) \\ &= 1 - 0.9938 \\ &= \boxed{0.0062} \end{aligned}$$

(b)

$$\begin{aligned} P(X < 88) &= P(Z < \frac{88 - 100}{16}) \\ &= P(Z < -0.75) \\ &= 1 - P(Z \leq 0.75) \\ &= 1 - 0.0.7734 \\ &= \boxed{0.2266} \end{aligned}$$

(c)

$$\begin{aligned} P(72 < X < 128) &= P(\frac{72 - 100}{16} < Z < \frac{128 - 100}{16}) \\ &= P(-1.75 < Z < 1.75) \\ &= P(Z \leq 1.75) - [1 - P(Z \leq 1.75)] \\ &= 0.9959 - (1 - 0.9959) \\ &= \boxed{0.9198} \end{aligned}$$

Question 3

An advertising agency conducted an ad campaign aimed at making consumers aware of a new product. Upon completion of the campaign, the agency claimed that 20% of consumers had become aware of the product (fixed 20% chance of a customer being aware of the product). Assume that each customer's chance of being aware of the product is independent. The product's distributor surveyed 1,000 consumers and found that 156 were aware of the product. Assume we believe the claim of the agency.

(a) Let X be the number of consumers who are aware of the product. What is the suitable probability distribution model for X?

(b) Calculate the mean μ and standard deviation σ of X.

(c) Find the probability that 156 or fewer consumers in a random sample of 1,000 would be aware of the product

Sample Solutions

(a) This is a binomial distribution.

$$X \sim N(n = 1000, p = 0.2)$$

(b)

$$\begin{aligned}\mu &= np = 1000 \times 0.2 = \boxed{200} \\ \sigma &= \sqrt{np(1-p)} = \sqrt{1000 \times 0.2 \times 0.8} = \boxed{12.6491}\end{aligned}$$

(c) Since $np = 200 > 5$ and $n(1-p) = 800 > 5$, we can use the normal approximation to the binomial distribution.

$$\begin{aligned}P(X \leq 156) &= P(Y < 156.5) \\ &= P(Z < \frac{156.5 - 200}{\sqrt{160}}) \\ &= P(Z < -3.44) \\ &= 1 - P(Z < 3.44) \\ &= 1 - 0.9997 \\ &= \boxed{0.0003}\end{aligned}$$

Question 4

The time, X minutes, taken by Nicholas to install a satellite dish may be assumed to be a normal random variable with mean 134 and standard deviation 16.

(a) Determine $P(X < 150)$

(b) Determine, the time exceeded by 10% of installations (only 10% of the installation times exceed this value).

(c) The time taken, Y minutes, taken by Nicholas to install a satellite dish may also be assumed to be normal random variable but with:

$$P(Y < 170) = 0.14 \text{ and } P(Y > 200) = 0.03$$

Determine the values for the mean and standard deviation of Y.

Sample Solutions

Given:
$\mu = 134$
$\sigma = 16$

(a)

$$\begin{aligned} P(X < 150) &= P(Z < \frac{150-134}{16}) \\ &= P(Z < 1) \\ &= \boxed{0.8413} \end{aligned}$$

(b) Let x be the time exceeded by 10% of installations.

$$\begin{aligned} P(X > x) &= 0.1 \\ &= 1 - P(X \leq x) \\ P(X \leq x) &= 0.9 \\ P(Z \leq \frac{x-134}{16}) &= 0.9 \\ \frac{x-134}{16} &= 1.28 \\ x &= 134 + 1.28 \times 16 = \boxed{154.48 \text{ minutes}} \end{aligned}$$

Question 4 continued on next page...

(c)

$$
\begin{aligned}
&\text{Equation 1:} \\
P(Y < 170) &= 0.14 \\
P(Z < \frac{170-\mu}{\sigma}) &= 0.14 \\
P(Z > \frac{170-\mu}{\sigma}) &= 1 - 0.14 = 0.86 \\
P(Z > 1.08) &= 0.86 \\
P(Z < -1.08) &= 1 - 0.86 = 0.14 \\
\frac{170-\mu}{\sigma} &= -1.08 \\
\mu - 1.08\sigma &= 170 \text{ Equation 1}
\end{aligned}
$$

$$
\begin{aligned}
P(Y > 200) &= 0.03 \\
P(Z > \frac{200-\mu}{\sigma}) &= 0.03 \\
P(Z < \frac{200-\mu}{\sigma}) &= 1 - 0.03 = 0.97 \\
P(Z < 1.88) &= 0.97 \\
\frac{200-\mu}{\sigma} &= 1.88 \\
\mu + 1.88\sigma &= 200 \text{ Equation 2}
\end{aligned}
$$

$$
\begin{aligned}
&\text{Equation 2 } - \text{Equation 1} \\
&30 = 2.96\sigma \\
&\sigma = \boxed{10.135} \\
&\mu = 170 + 1.08 \cdot (10.135) = \boxed{180.9462}
\end{aligned}
$$

Question 5

A strange new metal is discovered, and some coins are made from it. You're unsure if the properties of the metal will affect the fairness of the coins. However, based on some initial coin flips you think that 65% of the time the coins land on Tails, while 35% of the time it's Heads. Each coin flip is independent from the others and the probabilities are fixed. Assuming that we want to get as many Tails as possible for 1000 coin flips, answer the following:

(a) What type of probability distribution is suitable for this model?

(b) What is the mean and standard deviation for the 1000 coin flips?

(c) After the 1000 coin flips, 599 Tails are counted. Based on the current distribution, how likely is there to be 599 Tails or less?

Sample Solutions

Given:
$P(Tails) = 0.65$
$P(Heads) = 0.35$

(a)

$$X \sim B(n = 1000, p = 0.65)$$

(b)

$$\begin{aligned} \mu &= np = 1000 \times 0.65 = \boxed{650} \\ \sigma &= \sqrt{np(1-p)} \\ &= \sqrt{1000 \times 0.65 \times 0.35} \\ &= \boxed{15.0831} \end{aligned}$$

(c) $np > 5$ and $nq > 5$, therefore approximate to Normal $N(650, 15.0831^2)$

$$\begin{aligned} P(X \leq 599) &= P(Z < \frac{599.5 - 650}{15.0831}) \\ &= P(Z < -3.3481) \\ &= 1 - 0.9996 \\ &= \boxed{0.0004} \end{aligned}$$

Question 6

Based on past data, an insurance company determines that drivers in the age group of 21–28 have an 80% chance of no accidents in a year, a 20% chance of being in a single accident, and no chance of being in more than one accident in a year. For simplicity, assume that after an accident, there is a 50% probability that the car will need repairs costing \$500, a 40% probability of repairs costing \$5,000, and a 10% probability of repairs costing \$15,000.

(a) What is the expected repair cost?

(b) What is the potential variability of the repair cost; i.e., variance and standard deviation?

(c) Assume the insurer sells the policy to 100 car owners, each with the same risk, what is the insurer's total risk?

(d) To help cover the variability and other expenses, the insurer decided to charge an additional 30% over the expected repair cost. What would be the gross premium for a policy?

Sample Solutions

Given:
$P(0 \text{ accidents}) = 0.8$
$P(1 \text{ accident}) = 0.2$
$P(2 \text{ accidents}) = 0$
$P(500) = 0.5$
$P(5000) = 0.4$
$P(15000) = 0.1$

(a) Let X b the repair cost for an accident

x	0	500	5000	15000
$P(X = x)$	0.8	$0.5 \cdot 0.2 = 0.1$	$0.4 \cdot 0.2 = 0.08$	$0.1 \cdot 0.2 = 0.02$

$$
\begin{aligned}
E(X) = \mu = \sum xP(X = x) \\
&= 0 \times 0.8 + 500 \times 0.1 + 5000 \times 0.08 + 15000 \times 0.02 \\
&= \boxed{\$750}
\end{aligned}
$$

(b)

$$
\begin{aligned}
Var(x) = \sigma^2 = \sum (x-\mu)^2 P(X = x) \\
&= (0-750)^2 \times 0.8 + (500-750)^2 \times 0.1 + (5000-750)^2 \times 0.08 + (15000-750)^2 \times 0.02 \\
&= \boxed{\$5,962,500}
\end{aligned}
$$

$$\sigma = \sqrt{5962500} = \boxed{\$2442}$$

(c) Let X_1, X_2, ..., X_{100} be the repair cost for the 100 policies

$$\begin{aligned}
E(X_1 + X_2 + ... + X_{100}) &= 100 \times E(X) \\
&= 100 \times \$750 \\
&= \boxed{\$75,000} \\
Var(X_1 + X_2 + ... + X_{100}) &= 100 \times Var(X) \\
&= 100 \times \$5,962,500 \\
&= \boxed{\$596,250,000} \\
\text{Insurer's total risk} &= \text{coefficient of variation} \\
&= \frac{\sqrt{596250000}}{75000} \\
&= \boxed{0.326}
\end{aligned}$$

(d)

$$\begin{aligned}
\text{Gross premium} &= 1.3 \times E(X) \\
&= 1.3 \times \$750 \\
&= \boxed{\$975}
\end{aligned}$$

Part C (For your own thought):
Compare the insurer's risk if it only sold one policy, what do you observe?

$$\begin{aligned}
\text{Insurer's total risk} &= \text{coefficient of variation} \\
&= \frac{\sqrt{596250000}}{75000} \\
&= \boxed{0.326}
\end{aligned}$$

If the insurer only sold one policy, then

$$\begin{aligned}
\text{Insurer's total risk} &= \frac{\sqrt{596250000}}{750} \\
&= \boxed{3.26}
\end{aligned}$$

It is beneficial for insurer to sell more policies as it will reduce its total cost.

Question 7

A wholesaler sells a large number of 2-litre and 5-litre tubs of ice-cream which are sold in the ratio 3:2. A random sample of 2 tubs, $(X_1,\ X_2)$ is taken from the shelves.

(a) Find the mean and variance of the ice-cream content in this population.

(b) List all the possible samples.

(c) Find the sampling distribution for the mean $\bar{x}$.

My Solution

Given:
$P(2 \text{ litre}) = 0.6$
$P(5 \text{ litre}) = 0.4$

x	2	5
$P(X=x)$	0.6	0.4

(a)

$$\begin{aligned}\mu &= \sum xP(X=x)\\ &= 2\times 0.6 + 5\times 0.4\\ &= \boxed{3.2}\end{aligned}$$

$$\begin{aligned}\sigma^2 &= \sum (x-\mu)^2 P(X=x)\\ &= (2-3.2)^2\times 0.6 + (5-3.2)^2\times 0.4\\ &= \boxed{2.16}\end{aligned}$$

OR

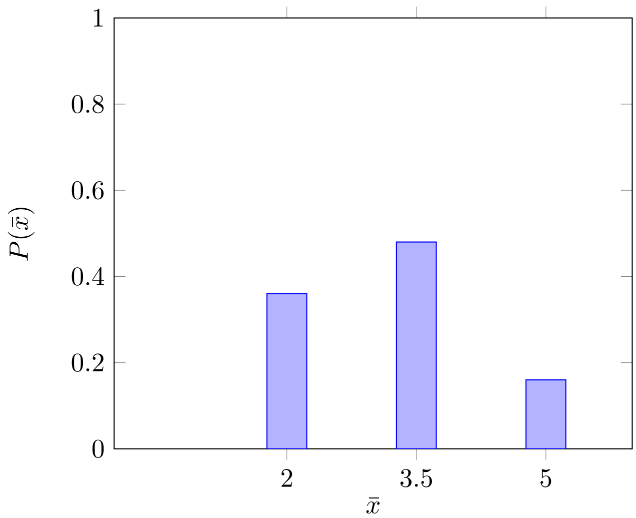
$$\begin{aligned}\sigma^2 &= \sum x^2 P(X=x) - \mu^2\\ &= (2^2\times 0.6 + 5^2\times 0.4) - 3.2^2\\ &= \boxed{2.16}\end{aligned}$$

(b) Total Possible combinations: ${}^2C_1 \times {}^2C_1 = 4$
Possible samples: $(2,2)$, $(2,5)$, $(5,2)$, $(5,5)$

(c)

Sample	$\bar{x}$	$P(\bar{x})$
$(2,2)$	2	$0.6\times 0.6 = 0.36$
$(2,5),(5,2)$	3.5	$0.6\times 0.4 + 0.4\times 0.6 = 0.48$
$(5,5)$	5	$0.4\times 0.4 = 0.16$

Question 7 continued on next page. . .



Aside

$$\begin{aligned}\text{Mean of sample means} &= \mu_{\bar{x}} = \sum \bar{x}P(\bar{x}) \\ &= 2 \times 0.36 + 3.5 \times 0.48 + 5 \times 0.16 \\ &= \boxed{3.2}\end{aligned}$$

$$\begin{aligned}\text{Variance of the Sample Means} &= \sigma_{\bar{x}}^2 = \sum (\bar{x} - \mu_{\bar{x}})^2 P(\bar{x}) \\ &= (2 - 3.2)^2 \times 0.36 + (3.5 - 3.2)^2 \times 0.48 + (5 - 3.2)^2 \times 0.16 \\ &= \boxed{1.08}\end{aligned}$$

Note that we get the following results:

$$\mu_x = \mu, \sigma_x^2 = \frac{\sigma^2}{n}$$

Question 8

A supermarket sells a large number of packets of bolts. One packet contains 10 bolts and the other 20 bolts. They are sold in the ratio 1:4 respectively. A random sample of 3 packets (X1, X2, X3) is taken from the shelves.

(a) List all the possible samples.

(b) Find the sampling distribution for the mode N and the median M

My Solution

Given:
$P(10 \text{ bolts}) = 0.2$
$P(20 \text{ bolts}) = 0.8$

X is the bolt packet being sold.

x	10	20
$P(X = x)$	0.2	0.8

(a)

$$\text{List all the possible samples: } \begin{matrix} (10,10,10) & & \\ (10,10,20) & (10,20,10) & (20,10,10) \\ (10,20,20) & (20,10,20) & (20,20,10) \\ (20,20,20) & & \end{matrix}$$

(b)

Sample	mode	P(mode)	median	P(median)
$(10,10,10)$	10	$(\frac{1}{3})^3$	10	$(\frac{1}{3})^3$
$(10,20,10)$ $(20,10,10)$ $(10,10,20)$	10	$3(\frac{1}{5})^2(\frac{2}{5})$	10	$3(\frac{1}{5})^2(\frac{2}{5})$
$(10,20,20)$ $(20,20,10)$ $(20,10,20)$	20	$3(\frac{4}{5})^2(\frac{1}{5})$	15	$3(\frac{4}{5})^2(\frac{1}{5})$
$(20,20,20)$	20	$(\frac{4}{5})^3$	20	$(\frac{4}{5})^3$

Sampling Distribution of the Sample Mode:

Mode	10	20
$P(\text{mode})$	$\frac{1}{125} + \frac{12}{125} = \frac{13}{125}$	$\frac{48}{125} + \frac{64}{125} = \frac{112}{125}$

Sampling Distribution of the Sample Median:

Median	10	20
$P(\text{median})$	$\frac{1}{125} + \frac{12}{125} = \frac{13}{125}$	$\frac{48}{125} + \frac{64}{125} = \frac{112}{125}$

Question 9

A random sample of 100 observations is to be drawn from a population with a mean of 40 and a standard deviation of 25. The population is a right-skewed distribution.

(a) Give the mean and standard deviation of the sampling distribution of mean of x.

(b) Will the sampling distribution of mean of x be approximately normal? Explain.

(c) Find the approximate probability that the mean of the sample will exceed 45.

My Solution

Given:
$\mu = 40$
$\sigma = 25$

(a)

$$\mu = \boxed{40}$$

$$\begin{aligned}\sigma &= \frac{\sigma}{\sqrt{n}}\\ &= \frac{25}{\sqrt{100}}\\ &= \boxed{2.5}\end{aligned}$$

(b) Yes. Even though we have the population distribution is right-skewed, since the sample size $n = 100$ is **large** ($n \geq 30$), we can still use the Central Limit Theorem to deduce that the sampling distribution of X is approximately normally distributed with mean 40 and standard deviation 2.5

(c) The Probability that the sample mean will exceed 45:

$$\begin{aligned}\bar{x} &\sim (40, 2.5^2)\\ Pr(\bar{x} > 45) &= Pr(\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} > \frac{45 - \mu}{\frac{\sigma}{\sqrt{n}}})\\ &= Pr(Z > \frac{45 - 40}{2.5})\\ &= P(Z > 2)\\ &= 1 - P(Z \leq 2)\\ &= 1 - 0.9772\\ &= \boxed{0.0228}\end{aligned}$$

Question 10

In an article in the Journal of American Pediatric Health researchers claim that the weights of healthy babies born in the United States form a distribution that is nearly Normal with an average weight of 7.25 pounds and standard deviation of 1.75 pounds. Suppose a researcher selects 50 random samples with 30 newborns in each sample.

(a) What is the best estimate for the mean of the sample means?

(b) What is the best estimate of the standard deviation of the sample means?

(c) If we randomly selected 30 newborns from the full population of US newborns, would you be surprised if their mean weight was 8.30 pounds?

My Solution

Given:
$\mu = 7.25$
$\sigma = 1.75$

$$X \sim N(7.25, 1.75^2)$$

(a)

$$\mu_{\bar{X}} = \mu = 7.25$$

(b)

$$\begin{aligned} \sigma_{\bar{X}} &= \frac{\sigma}{\sqrt{n}} \\ &= \frac{1.75}{\sqrt{30}} \\ &= 0.31950 \\ &\approx \boxed{0.32} \end{aligned}$$

(c) Yes. The sample mean of 8.30 is more than 3 standard deviations away from the population mean of 7.25.

$$3 \text{ standard deviations away} = 7.5 + 3(\frac{1.75}{\sqrt{30}}) = \boxed{8.21}$$

Answers:
Yes, a mean of 8.30 pounds would be surprising as this sample result is more than 3 standard deviations above the overall mean weight of 7.25.
Alternatively, we find that

$$P(\bar{X} \geq 8.30) = P(Z > \frac{8.30 - 7.25}{0.32}) = P(Z > 3.28) = 1 - P(Z \leq 3.28) = 1 - 0.9995 = 0.0005$$

Question 11

The length of human gestation is well-approximated by a normal distribution with mean $\mu = 280$ days and standard deviation $\sigma = 8.5$ days. Suppose your final exam is scheduled for May 2 and your pregnant lady teacher has a due date of May 9. Find the probability that she will give birth on or before the day of the final exam (0.2061).

My Solution

Given:

$\mu = 280$ days
$\sigma = 8.5$ days
Let X be the number of days until the baby is born before the final exam.

$$X \sim N(280, 8.5^2)$$

$$\begin{aligned}
P(X \leq 273) &= P(Z \leq \frac{273 - 280}{8.5}) \\
&= P(Z \leq -0.8235) \\
&= 1 - P(Z < 0.8235) \\
&= 1 - 0.7939 \\
&= \boxed{0.2061}
\end{aligned}$$

Question 12

Question 8 (Modified)
A supermarket sells a large number of packets of bolts. One packet contains 10 bolts and the other 20 bolts. They are sold in the ratio 1:4 respectively. A random sample of 3 packets (X1, X2, X3) is taken from the shelves.

(a) Find the sampling distribution of the sample mean.

(b) Compare the mean and standard deviation of the sampling distribution of the sample mean with the population mean and population standard deviation.

Sample Solutions

Sample	mean	P(mean)
$(10, 10, 10)$	10	$(\frac{1}{5})^3 = \frac{1}{125}$
$(10, 20, 10)$ $(20, 10, 10)$ $(10, 20, 20)$	10	$(\frac{1}{5}^2)(\frac{4}{5}) = \frac{12}{125}$
$(10, 20, 20)$ $(20, 20, 10)$ $(20, 10, 20)$	20	$(\frac{4}{5}^2)(\frac{1}{5}) = \frac{48}{125}$
$(20, 20, 20)$	20	$(\frac{4}{5})^3 = \frac{64}{125}$

$$\begin{aligned}
\mu_x &= 10 \cdot \frac{1}{125} + 403(\frac{48}{125}) + 20(\frac{64}{125}) \\
&= \frac{2250}{125} \\
&= \boxed{18}
\end{aligned}$$

$$\begin{aligned}
\sigma_{\bar{x}}^2 &= (10 - 18)^2(\frac{1}{125}) + (\frac{40}{3} - 18)^2(\frac{12}{125}) + (\frac{50}{3} - 18)^2(\frac{48}{125}) + (20 - 18)^2(\frac{64}{125}) \\
&= \boxed{\frac{16}{3}} = \boxed{5.33}
\end{aligned}$$

As a comparison to probability distribution of X:

x	10	20
P(x)	$\frac{1}{5}$	$\frac{4}{5}$

$$\begin{aligned}
\mu &= 10(\frac{1}{5}) + 20(\frac{4}{5}) \\
&= \boxed{18} \text{ thus } \mu_{\bar{x}} = \mu \\
\sigma^2 &= (10 - 18)^2(\frac{1}{5}) + (20 - 18)^2(\frac{4}{5}) \\
&= \boxed{16} \text{ thus } \sigma_{\bar{x}}^2 = \frac{\sigma^2}{n} \\
\sigma &= 4 \\
\sigma_{\bar{x}} &= \frac{4}{\sqrt{3}}
\end{aligned}$$

Question 12 continued on next page...

Question 12

The average male drinks 2 litres of water when active outdoors with the standard deviation of 0.7 litres. You are planning a full day nature trup for 50 men and will bring 110 lires of water.
What is the probability that you will have enough water? (0.9783)
How much water must you bring to be 99% confident of having enough water? (111.5)

My Solution

P(enough Water) = P(50 men drink at most 110 litres)
= P(each person drinks on average at most 2.2 litres)
$= P(\bar{x} \leq 2.2)$

$$\bar{x} \sim N(2, \frac{0.7^2}{50})$$

$$\begin{aligned}
P(\bar{x} \leq 2.2) &= P(Z \leq \frac{2.2 - 2}{\sqrt{\frac{0.7^2}{50}}}) \\
&= P(Z \leq 2.02) \\
&= \boxed{0.9783}
\end{aligned}$$

$$\begin{aligned}
P(\bar{x} < \alpha) &= 0.99 \text{since P(Z ¡ 2.33) = 0.99} \\
\frac{\alpha - 2}{\sqrt{\frac{0.7}{50}}} &= 2.33 \\
\alpha &= 2.2307 \\
50\alpha &= 111.5
\end{aligned}$$

Question 12 continued on next page. . .