INF 1004 Mathematics 2
Revision Tutorial Solutions

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Question 1

Remove vector $u = (-1, 3, -4, 2)$ from vector $v = (-2, 2, 2.5, 6)$

Sample Solutions

$$v_{new} = v - \frac{v \cdot u}{u \cdot u} u$$

$$u \cdot u = (-1, 3, -4, 2) \cdot (-1, 3, -4, 2) = 30$$
$$v \cdot u = (-2, 2, 2.5, 6) \cdot (-1, 3, -4, 2) = 10$$

$$\begin{aligned} v_{new} &= (-2, 2, 2.5, 6) - \frac{10}{30}(-1, 3, -4, 2) \\ &= (-2, 2, 2.5, 6) + (\frac{1}{3}, -1, \frac{4}{3}, -\frac{2}{3}) \\ &= (-\frac{5}{3}, 1, \frac{13}{6}, \frac{16}{3}) \end{aligned}$$

You can check:

$$v_{new} \cdot u = (-\frac{5}{3}, 1, \frac{13}{6}, \frac{16}{3}) \cdot (-1, 3, -4, 2) = 0$$

Question 2

Note the difference in the order between remove from and project onto.
Project vector $u = (-1, -3, -4, 2)$ onto vector $v = (3, -3, -1, 1)$

Sample Solutions

$$u_{new} = \frac{u \cdot v}{v \cdot v} v$$

$$\begin{aligned} u_{new} &= \frac{(-1, -3, -4, 2) \cdot (3, -3, -1, 1)}{(3, -3, -1, 1) \cdot (3, -3, -1, 1)} (3, -3, -1, 1) \\ &= \frac{12}{20} (3, -3, -1, 1) \end{aligned}$$

Question 3

$$x + 4y + 2z = 5.5$$
$$-5x - 22y - 5z = -45.5$$
$$2x + 4y + 14z = -25$$

- Show as an itermediate step the augmented matrix when for the first time the zeroth coulmn became a one-hot vector after performing transformations
- Show as an intermediate step the augmented matrix when for the first time the augmented matrix is in row echelon form.
- Write the set of all solutions as a single vector or a combination of vectors, None if there is no solution

My Solution

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 5.5 \\ -5 & -22 & -5 & -45.5 \\ 2 & 4 & 14 & -25 \end{array}\right]$$

$$\rho_2 + 5\rho_1, \rho_3 - 2\rho_1 \rightarrow \left[\begin{array}{ccc|c} 1 & 4 & 2 & 5.5 \\ 0 & -2 & 5 & -18 \\ 0 & -4 & 10 & -36 \end{array}\right]$$

$$\rho_3 - 2\rho_2 \rightarrow \left[\begin{array}{ccc|c} 1 & 4 & 2 & 5.5 \\ 0 & -2 & 5 & -18 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$-\frac{1}{2}\rho_2 \rightarrow \left[\begin{array}{ccc|c} 1 & 4 & 2 & 5.5 \\ 0 & 1 & -2.5 & 9 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\rho_1 + 4\rho_2 \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 12 & -30.5 \\ 0 & 1 & -2.5 & 9 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x + 12z = -30.5$$
$$y - 2.5z = 9$$

$$\therefore x = -30.5 - 12z$$
$$\therefore y = 9 + 2.5z$$
$$\therefore z = 0 + 1z$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -30.5 \\ 9 \\ 0 \end{bmatrix} + \begin{bmatrix} -12 \\ 2.5 \\ 1 \end{bmatrix} z$$

Question 4

$$x + 3y - 5z = 2.75$$
$$3x + 12y - 13z = -9.75$$
$$-4x - 6y + 25z = -46.25$$

- Show as an itermediate step the augmented matrix when for the first time the zeroth coulmn became a one-hot vector after performing transformations
- Show as an intermediate step the augmented matrix when for the first time the augmented matrix is in row echelon form.
- Write the set of all solutions as a single vector or a combination of vectors, None if there is no solution

Sample Solutions

$$\left[\begin{array}{ccc|c} 1 & 3 & -5 & 2.75 \\ 3 & 12 & -13 & -9.75 \\ -4 & -6 & 25 & -46.25 \end{array}\right]$$

$$\rho_2 - 3\rho_1, \rho_3 + 4\rho_1 \rightarrow \left[\begin{array}{ccc|c} 1 & 3 & -5 & 2.75 \\ 0 & 3 & 2 & -18 \\ 0 & 6 & 5 & -35.25 \end{array}\right]$$

$$\rho_3 - 2\rho_2 \rightarrow \left[\begin{array}{ccc|c} 1 & 3 & -5 & 2.75 \\ 0 & 3 & 2 & -18 \\ 0 & 0 & 1 & 0.75 \end{array}\right]$$

$$\frac{1}{3}\rho_2 - 2\rho_1 \rightarrow \left[\begin{array}{ccc|c} 1 & 3 & -5 & 2.75 \\ 0 & 1 & 0 & -6.5 \\ 0 & 0 & 1 & 0.75 \end{array}\right]$$

$$\rho_1 - 3\rho_2 + 5\rho_1 \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 26 \\ 0 & 1 & 0 & -6.5 \\ 0 & 0 & 1 & 0.75 \end{array}\right]$$

$$\text{first time one hot:} \left[\begin{array}{ccc|c} 1 & 3 & -5 & 2.75 \\ 0 & 3 & 2 & -18 \\ 0 & 6 & 5 & -35.25 \end{array}\right]$$

$$\text{first time row echelon form:} \left[\begin{array}{ccc|c} 1 & 3 & -5 & 2.75 \\ 0 & 1 & \frac{2}{3} & 6 \\ 0 & 0 & 1 & 0.75 \end{array}\right]$$

$$\text{Reduced row echelon form:} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 26 \\ 0 & 1 & 0 & -6.5 \\ 0 & 0 & 1 & 0.75 \end{array}\right]$$

Question 5

Compute the inverse of

$$A_0 = \begin{bmatrix} 9 & -2 \\ 3 & -4 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 10 & 3 \\ 8 & 4 \end{bmatrix}$$

Use these inverses to Solve

$$A_0 x = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$A_1 x = \begin{bmatrix} -7 \\ 4 \end{bmatrix}$$

Sample Solutions

$$\begin{aligned} A_0 &= \begin{bmatrix} 9 & -2 \\ 3 & -4 \end{bmatrix} \\ \det(A_0) &= 9 \cdot (-4) - (-2) \cdot 3 = -36 + 6 = -30 \\ A_0^{-1} &= \frac{1}{-30} \begin{bmatrix} -4 & 2 \\ -3 & -9 \end{bmatrix} \\ &= \begin{bmatrix} \frac{2}{15} & -\frac{1}{15} \\ \frac{1}{10} & -\frac{3}{10} \end{bmatrix} \\ x &= A_0^{-1} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{2}{15} & -\frac{1}{15} \\ \frac{1}{10} & -\frac{3}{10} \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{2}{15} + \frac{2}{15} \\ \frac{1}{10} + \frac{6}{10} \end{bmatrix} \\ &= \begin{bmatrix} \frac{4}{15} \\ \frac{7}{10} \end{bmatrix} \end{aligned}$$

$$\begin{aligned} A_1 &= \begin{bmatrix} 10 & 3 \\ 8 & 4 \end{bmatrix} \\ \det(A_1) &= 10 \cdot 4 - 3 \cdot 8 = 40 - 24 = 16 \\ A_1^{-1} &= \frac{1}{16} \begin{bmatrix} 4 & -3 \\ -8 & 10 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{4} & -\frac{3}{16} \\ \frac{-1}{2} & \frac{5}{8} \end{bmatrix} \\ x &= \begin{bmatrix} \frac{1}{4} & -\frac{3}{16} \\ \frac{-1}{2} & \frac{5}{8} \end{bmatrix} \begin{bmatrix} -7 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{10}{4} \\ 6 \end{bmatrix} \end{aligned}$$

Question 6

Compute the determinant of

$$A = \begin{bmatrix} 3 & -1 & 4 \\ 5 & 2.5 & 3 \\ 1 & 8 & -6 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & -2 & 0.5 \\ 2.5 & -3 & 1 \\ 3 & 2 & 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -2 & 2 \\ 8 & 3 & -2 \\ 10 & -4.5 & 5 \end{bmatrix}$$

- Are they invertible?
- Which of them has full rank? Which one of them has lower rank and which one?

Sample Solutions

$$A = \begin{bmatrix} 3 & -1 & 4 \\ 5 & 2.5 & 3 \\ 1 & 8 & -6 \end{bmatrix}$$

$\det A = 0$(not invertible)

$\text{rank} = 2$

$$A = \begin{bmatrix} 3 & -2 & 0.5 \\ 2.5 & -3 & 1 \\ 3 & 2 & 4 \end{bmatrix}$$

$\det A = -21$(invertible)

$\text{rank} = 3$

$$A = \begin{bmatrix} 2 & -2 & 2 \\ 8 & 3 & -2 \\ 10 & -4.5 & 5 \end{bmatrix}$$

$\det A = 0$(not invertible)

$\text{rank} = 2$

Question 7

What is the determinant of this matrix? Write it as a polynomial in c.
For what value c the matrix is not invertible?

$$A = \begin{bmatrix} 6 & -3 & c \\ 5 & 2 & 2 \\ -2 & -6 & -2 \end{bmatrix}$$

Sample Solutions

$$\begin{aligned} \det(A) &= 30 - 30c + 4c \\ &= 30 - 26c \\ \det(A) &= 0 \\ \therefore c &= \frac{30}{26} \end{aligned}$$

Question 8

Compute and apply the Householder matrix which makes transforms the first column of A to a multipile of the first one-hot vector for

$$A = \begin{bmatrix} 8 & 1 & 2 \\ 4 & -1 & 3 \\ -8 & 4 & 2 \end{bmatrix}$$

and for (Subtracting is nicer)

$$A = \begin{bmatrix} 3 & -4 & 3 \\ \sqrt{2} & 6 & 4 \\ \sqrt{5} & 3 & 2 \end{bmatrix}$$

Sample Solutions

Part 1

$$x = \begin{bmatrix} 8 \\ 4 \\ -8 \end{bmatrix}$$

$$u = \begin{bmatrix} 8 \\ 4 \\ -8 \end{bmatrix} \pm \| [8, 4, -8] \| \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -4 \\ 4 \\ -8 \end{bmatrix} \text{ Subtract}$$

$$H_u = I - \frac{2}{\|[-4, 4, -8]\|_2^2} \begin{bmatrix} 8 \\ 4 \\ -8 \end{bmatrix} \begin{bmatrix} 8 & 4 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

$$HA = \begin{bmatrix} 12 & -2.44 & 1 \\ 0 & 2.33 & 4 \\ 0 & -2.7 & 0 \end{bmatrix}$$

Part 2

$$x = \begin{bmatrix} 3 \\ \sqrt{2} \\ \sqrt{5} \end{bmatrix}$$

$$u = \begin{bmatrix} 3 \\ \sqrt{2} \\ \sqrt{5} \end{bmatrix} \pm \| [3, \sqrt{2}, \sqrt{5}] \| \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ \sqrt{2} \\ \sqrt{5} \end{bmatrix} \text{ Subtract}$$

$$H_u = I - \frac{2}{\|[-1, \sqrt{2}, \sqrt{5}]\|_2^2} \begin{bmatrix} -1 \\ \sqrt{2} \\ \sqrt{5} \end{bmatrix} \begin{bmatrix} -1 & \sqrt{2} & \sqrt{5} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{4} & \frac{\sqrt{2}}{4} & \frac{\sqrt{5}}{4} \\ \frac{\sqrt{2}}{4} & \frac{1}{2} & -\frac{\sqrt{2}\sqrt{5}}{4} \\ \frac{\sqrt{5}}{4} & -\frac{\sqrt{5}\sqrt{2}}{4} & -\frac{1}{4} \end{bmatrix}$$

$$HA = \begin{bmatrix} 4 & 0.8 & 4.78 \\ 0 & -0.79 & 1.48 \\ 0 & -7.73 & -1.99 \end{bmatrix}$$