INF 1004 Mathematics 2
Tutorial #11

Woon Jun Wei
2200624

March 24, 2023

Question 1

- Compute the eigenvalues and eigenvectors for the matrices below.
- For one of these matrices compute the matrix P such that $P^{-1}DP = A$, and verify that PAP^{-1} is the diagonal matrix of the eigenvalues

$$A = \begin{bmatrix} 4 & \sqrt{15} \\ \sqrt{15} & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -2 \\ -3 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -4 \\ 4 & -6 \end{bmatrix}$$

My Solution

Question 1 continued on next page...

Part 1

$$A = \begin{bmatrix} 4 & \sqrt{15} \\ \sqrt{15} & 2 \end{bmatrix}$$

$$\det \begin{bmatrix} 4-\lambda & \sqrt{15} \\ \sqrt{15} & 2-\lambda \end{bmatrix} = 0$$

$$\begin{aligned} f(\lambda) = \lambda^2 - 6\lambda - 7 = 0 \\ (\lambda - 7)(\lambda + 1) = 0 \\ \lambda = 7, -1 \end{aligned}$$

$$\begin{aligned} \lambda_1 &= -1, \\ A - (-1)I &= \begin{bmatrix} 5 & \sqrt{15} \\ \sqrt{15} & 3 \end{bmatrix} \\ &= \begin{bmatrix} \sqrt{15} & 3 \\ 0 & 0 \end{bmatrix} \\ \sqrt{15}x_0 + 3x_1 &= 0 \\ x_0 &= -\frac{3}{\sqrt{15}}x_1 \\ \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} &= x_1 \begin{bmatrix} -\frac{3}{\sqrt{15}} \\ 1 \end{bmatrix} \\ \therefore v_1 &= \begin{bmatrix} -3 \\ \sqrt{15} \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \lambda_1 &= 7, \\ A - (7)I &= \begin{bmatrix} -3 & \sqrt{15} \\ \sqrt{15} & -5 \end{bmatrix} \\ &= \begin{bmatrix} \sqrt{15} & -5 \\ 0 & 0 \end{bmatrix} \\ \sqrt{15}x_0 - 5x_1 &= 0 \\ x_0 &= \frac{5}{\sqrt{15}}x_1 \\ \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} &= x_1 \begin{bmatrix} \frac{5}{\sqrt{15}} \\ 1 \end{bmatrix} \\ \therefore v_1 &= \begin{bmatrix} 5 \\ \sqrt{15} \end{bmatrix} \end{aligned}$$

Case: Symmetric Matix, 2 different eigenvalues, 2 eigenvectors
Eigenspaces are orthogonal, Check: $v_{01} = -15 + \sqrt{15}\sqrt{15} = 0$

Eigendecomposition

$\lambda_1 = -1, v_1 = \begin{bmatrix} -3 \\ \sqrt{15} \end{bmatrix}, \qquad \lambda_2 = 7, v_2 = \begin{bmatrix} 5 \\ \sqrt{15} \end{bmatrix}$

$p^{-1} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} -3 & 5 \\ \sqrt{15} & \sqrt{15} \end{bmatrix}$

$D = \begin{bmatrix} -1 & 0 \\ 0 & 7 \end{bmatrix}$

$p = (p^{-1})^{-1} = \frac{1}{-3\sqrt{15}-5\sqrt{15}} \begin{bmatrix} \sqrt{15} & -5 \\ -\sqrt{15} & -3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{8} & \frac{5}{8\sqrt{15}} \\ \frac{1}{8} & \frac{3}{8\sqrt{15}} \end{bmatrix}$

Verify:

$$P^{-1}DP = \begin{bmatrix} -3 & 5 \\ \sqrt{15} & \sqrt{15} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} -\frac{1}{8} & \frac{5}{8\sqrt{15}} \\ \frac{1}{8} & \frac{3}{8\sqrt{15}} \end{bmatrix} = \begin{bmatrix} 4 & \sqrt{15} \\ \sqrt{15} & 2 \end{bmatrix} = A$$

$$PAP^{-1} = \begin{bmatrix} -\frac{1}{8} & \frac{5}{8\sqrt{15}} \\ \frac{1}{8} & \frac{3}{8\sqrt{15}} \end{bmatrix} \begin{bmatrix} 4 & \sqrt{15} \\ \sqrt{15} & 2 \end{bmatrix} \begin{bmatrix} -\frac{1}{8} & \frac{5}{8\sqrt{15}} \\ \frac{1}{8} & \frac{3}{8\sqrt{15}} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 7 \end{bmatrix} = D$$

Question 1 continued on next page. . .

Part 2

$$A = \begin{bmatrix} 2 & -2 \\ -3 & 1 \end{bmatrix}$$

$$\det \begin{bmatrix} 2-\lambda & -2 \\ -3 & 1-\lambda \end{bmatrix} = 0$$

$$\begin{aligned} f(\lambda) = \lambda^2 - 3\lambda - 4 &= 0 \\ (\lambda - 4)(\lambda + 1) &= 0 \\ \therefore \lambda = 4, -1 \end{aligned}$$

$$\begin{aligned} \lambda_1 &= -1, \\ A - (-1)I &= \begin{bmatrix} 3 & -2 \\ -3 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -2 \\ 0 & 0 \end{bmatrix} \\ 3x_0 - 2x_1 &= 0 \\ x_0 &= \frac{2}{3}x_1 \\ \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} &= x_1 \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} \\ \therefore v_1 &= \begin{bmatrix} 2 \\ 3 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \lambda_2 &= 4, \\ A - (4)I &= \begin{bmatrix} -2 & -2 \\ -3 & -3 \end{bmatrix} \\ &= \begin{bmatrix} -2 & -2 \\ 0 & 0 \end{bmatrix} \\ -2x_0 - 2x_1 &= 0 \\ x_0 &= -x_1 \\ \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} &= x_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ \therefore v_2 &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{aligned}$$

Eigenspaces are not orthogonal, but matrix is not symetric
So even though we get 2 eigenvalues and 2 eigenvectors, it is okay that the eigenvectors are not orthogonal to each other.

Eigendecomposition

$\lambda_1 = -1, v_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \qquad \lambda_2 = 4, v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$p^{-1} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$

$D = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$

$p = (p^{-1})^{-1} = \frac{1}{-2-3} \begin{bmatrix} -1 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{3}{5} & -\frac{2}{5} \end{bmatrix}$

$$\begin{aligned} P^{-1}DP &= \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{3}{5} & -\frac{2}{5} \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -3 & 1 \end{bmatrix} = A \\ PAP^{-1} &= \begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{3}{5} & -\frac{2}{5} \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} = D \end{aligned}$$

Part 3

$$A = \begin{bmatrix} 2 & -4 \\ 4 & -6 \end{bmatrix}$$

$$\det \begin{bmatrix} 2-\lambda & -4 \\ 4 & -6-\lambda \end{bmatrix} = 0$$

$$\begin{aligned} f(\lambda) = \lambda^2 + 4\lambda + 4 = 0 \\ (\lambda+2)(\lambda+2) = 0 \\ \lambda = -2, -2 \end{aligned}$$

$$\begin{aligned} \lambda_1 &= -2, \\ A - (-2)I &= \begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix} \\ &= \begin{bmatrix} 4 & -4 \\ 0 & 0 \end{bmatrix} \\ 4x_0 - 4x_1 &= 0 \\ x_0 &= x_1 \\ \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} &= x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \therefore v_1 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

Eigenspace just one dim, but matrix is not symmetric
So it is okay that we do not have 2 eigenvalues nor 2 eigenvectors. (We cant compute D since there is only 1 eigenvector)

Question 2

- Compute an eigenvector for the eigenvalue $x = 3$ for the below $(3, 3)$-matrix. Note: nobody asks you to compute its characteristic polynomial or to get all of its eigenvalues (Prof did it)
- Validate that the found eigenvector v is indeed the correct one, that is, that $Av = 3v$ holds.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 2 & 4 & 4 \end{bmatrix}$$

My Solution

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & -1 \\ 2 & 4 & 4-\lambda \end{bmatrix}$$

$$f\lambda = -\lambda^3 + +6\lambda^2 - 11\lambda + 6 = 0$$

$$f(3) = 3^3 - 6 \cdot 3^2 + 11 \cdot 3 + 6 = 0$$

When $\lambda = 3$,

$$A - 3I = \begin{bmatrix} -2 & 0 & 1 \\ 0 & -2 & -1 \\ 2 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

1^{st} row: $-2x_0 + x_2 = 0 \Rightarrow x_0 = \frac{1}{2}x_2$
2^{nd} row: $-2x_1 - x_2 = 0 \Rightarrow x_1 = -\frac{1}{2}x_2$

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

$$\therefore v = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Verify:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 2 & 4 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Question 3

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

- Show that this matrix has only the eigenvalue 1, twice.
- Prove that there cannot exist any matrix P such that $P^{-1}DP = A$. Hint: You know how D in $P^{-1}DP$ must look like
- Find an eigenvector

Bonus knowldge: Shear matrices have an eigenspace of dimensionality $d-1$. In the above case the set of all eigenvectors must be $cv, c \in \mathbb{R}$ for some vector v.

My Solution

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix}$$

$$f(\lambda) = (\lambda - 1)^2 = 0$$

So only $\lambda = 1$ is a solution. It has 2 eigenvalues equal to 1 (*eigenvalue '1' has algebraic mutiplicity of 2*)

Suppose P exists such that $P^{-1}DP = A$, then $D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
$P^{-1}DP = P^{-1}IP = P^{-1}P = I$
But A is not the identity matrix. So there cannot exist any P as assumed above (Proof by Contradiction)

$$A - I = \begin{bmatrix} o & 1 \\ 0 & 0 \end{bmatrix}$$

$$0 \cdot x_0 + 1 \cdot x_1 = 0 \Rightarrow x_1 = 0$$

$$\begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\therefore \text{ eigenvector } v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Question 4

Show that

$$\begin{bmatrix} 2 & -4 \\ \frac{13}{4} & -4 \end{bmatrix}$$

- has no real eigenvalue
- Bonus: what are its complex-valued eigenvalues?

My Solution

$$A - \lambda I = \begin{bmatrix} 2-\lambda & -4 \\ \frac{13}{4} & -4-\lambda \end{bmatrix}$$

$$f(\lambda) = (\lambda - 2)(\lambda + 4) = \lambda^2 + 2\lambda + 5 = 0$$

$$\lambda = \frac{-(2) \pm \sqrt{(2)^2 - 4 \cdot 5}}{2} = \frac{-2 \pm 2\sqrt{5}}{2} = \frac{-2 \pm \sqrt{-1}\sqrt{16}}{2} = \frac{-2 \pm i(4)}{2} = -1 \pm 2i$$

There are no real eigenvalues.

Question 5

Bonus Matrix:

- Get its eigenvalues and eigenvectors

Note: This is a symmetric one, so you can expect 2 eigenvalues and orthogonal eigenspace

$$A = \begin{bmatrix} -2 & \sqrt{24} \\ \sqrt{24} & 8 \end{bmatrix}$$

My Solution

$$A = \begin{bmatrix} -2 & \sqrt{24} \\ \sqrt{24} & 8 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} -2-\lambda & \sqrt{24} \\ \sqrt{24} & 8-\lambda \end{bmatrix}$$

$$f(\lambda) = (\lambda+2)(\lambda-8) = \lambda^2 - 6\lambda - 40 = (\lambda-10)(\lambda+4) = 0$$

So $\lambda = 10$ or $\lambda = -4$

$$\begin{aligned} A - -4I &= \begin{bmatrix} 2 & \sqrt{24} \\ \sqrt{24} & 12 \end{bmatrix} \\ &= \begin{bmatrix} \sqrt{24} & 12 \\ 0 & 0 \end{bmatrix} \\ \sqrt{24}x_0 + 12x_1 &= 0 \\ x_0 &= -\frac{12}{\sqrt{24}}x_1 \\ \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} &= \begin{bmatrix} -\frac{12}{\sqrt{24}} \\ 1 \end{bmatrix} \\ \therefore v_1 &= \begin{bmatrix} -12 \\ \sqrt{24} \end{bmatrix} \end{aligned}$$

$$\begin{aligned} A - 10I &= \begin{bmatrix} -12 & \sqrt{24} \\ \sqrt{24} & -2 \end{bmatrix} \\ &= \begin{bmatrix} -12 & \sqrt{24} \\ 0 & 0 \end{bmatrix} \\ -12x_0 + \sqrt{24}x_1 &= 0 \\ x_0 &= \frac{\sqrt{24}}{12}x_1 \\ \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} &= \begin{bmatrix} \frac{\sqrt{24}}{12} \\ 1 \end{bmatrix} \\ \therefore v_2 &= \begin{bmatrix} \sqrt{24} \\ 12 \end{bmatrix} \end{aligned}$$

Question 6

- Use numpy to get its eigenvalues and eigenvectors
- Solve $Ax = (3, 17, 1/3)$ using numpy
- Not in exam:

Compute the characteristic polynomial for

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & -2 & 1 \\ 3 & 1 & 3 \end{bmatrix}$$

My Solution

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 2 & 4 \\ 2 & -2-\lambda & 1 \\ 3 & 1 & 3-\lambda \end{bmatrix}$$

$$f(\lambda) = -\lambda^3 + 2\lambda^2 + 22\lambda + 19$$

Question 7

What is the cosine of the angle between

$$(6, -6, -4, \sqrt{12}), (6, 4, 2, \sqrt{25})$$

My Solution

Question 8

Another 3x3 affine system

- Show the intermediate result when the first column is the one hot vector $\begin{bmatrix}1\\0\\0\end{bmatrix}$ for the first time
- Show the intermediate result when the matrix has row echelon form for the first time
- Get the solution

$$\begin{aligned} 2x - 3y + 2z &= -4 \\ 7x + 4.5y - 1z &= 16 \\ 4x + 3y + z &= 2 \end{aligned}$$

My Solution